Introduction to Vector Spaces and Subspaces – Sections 4.1-4.2, 4.7

Math 81, Applied Analysis
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1 Introduction to Vector Spaces – Sections 4.1-4.2, 4.7

1.1 Vector Spaces

Definition: Let V be a set of elements called vectors, in which the operations of addition of vectors and multiplication of vectors by scalars are defined. Then, given any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars a and b, V is a **vector space** if

(1) $\mathbf{u} + \mathbf{v} \in V$ (V is closed under vector addition)

(2) $a\mathbf{u} \in V$ (V is closed under scalar multiplication)

(3) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

(4) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity)

(5) There exists a zero element in V, $\mathbf{0} \in V$, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(6) There exists an element in V, $-\mathbf{u}$, called the additive inverse, such that

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

(7) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

(8) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

(9) $a(b\mathbf{u}) = (ab)\mathbf{u}$

(10) $(1)\mathbf{u} = \mathbf{u}$

Example: $\mathbb{R}^3$ with the standard vector addition and scalar multiplication is a vector space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and a, b scalars. So,

$$\begin{aligned}\mathbf{u} &= (u_1, u_2, u_3)\\ \mathbf{v} &= (v_1, v_2, v_3)\\ \mathbf{w} &= (w_1, w_2, w_3)\end{aligned}$$

Verify that $\mathbb{R}^3$ satisfies the properties for a vector space:

(1) $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,u_3+v_3)\in\mathbb{R}^3$ $\surd$

(2) $a\mathbf{u}=(au_1,au_2,au_3)\in\mathbb{R}^3$ $\surd$

(3) Commutativity:

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(u_1+v_1,u_2+v_2,u_3+v_3)\\&=(v_1+u_1,v_2+u_2,v_3+u_3)\\&=\mathbf{v}+\mathbf{u}\ \surd\end{aligned}$$

(4) Associativity:

$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w})&=(u_1,u_2,u_3)+(v_1+w_1,v_2+w_2,v_3+w_3)\\&=(u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3)\\&=((u_1+v_1)+w_1,(u_2+v_2)+w_2,(u_3+v_3)+w_3)\\&=(u_1+v_1,u_2+v_2,u_3+v_3)+(w_1,w_2,w_3)\\&=(\mathbf{u}+\mathbf{v})+\mathbf{w}\ \surd\end{aligned}$$

(5) $\mathbf{0}=(0,0,0)$
$\mathbf{u}+\mathbf{0}=(u_1+0,u_2+0,u_3+0)=(u_1,u_2,u_3)=\mathbf{u}$ $\surd$

(6) $\mathbf{u}=(u_1,u_2,u_3)\rightarrow-\mathbf{u}=(-u_1,-u_2,-u_3)$
$\mathbf{u}+(-\mathbf{u})=(u_1+(-u_1),u_2+(-u_2),u_3+(-u_3))=(0,0,0)=\mathbf{0}$ $\surd$

(7)

$$\begin{aligned}a(\mathbf{u}+\mathbf{v})&=a\left(u_1+v_1,u_2+v_2,u_3+v_3\right)\\&=\left(a\left(u_1+v_1\right),a\left(u_2+v_2\right),a\left(u_3+v_3\right)\right)\\&=(au_1+av_1,au_2+av_2,au_3+av_3)\\&=(au_1,au_2,au_3)+(av_1,av_2,av_3)\\&=a\mathbf{u}+a\mathbf{v}\ \surd\end{aligned}$$

(8)

$$\begin{aligned}(a+b)\mathbf{u}&=(a+b)\left(u_1,u_2,u_3\right)\\&=((a+b)u_1,(a+b)u_2,(a+b)u_3)\\&=(au_1+bu_1,au_2+bu_2,au_3+bu_3)\\&=(au_1,au_2,au_3)+(bu_1,bu_2,bu_3)\\&=a\mathbf{u}+b\mathbf{u}\ \surd\end{aligned}$$

(9)

$$\begin{aligned}a(b\mathbf{u})&=a\left(bu_1,bu_2,bu_3\right)\\&=(abu_1,abu_2,abu_3)\\&=((ab)u_1,(ab)u_2,(ab)u_3)\\&=(ab)\mathbf{u}\ \surd\end{aligned}$$

(10) $(1)\mathbf{u}=1\left(u_1,u_2,u_3\right)=(1\cdot u_1,1\cdot u_2,1\cdot u_3)=(u_1,u_2,u_3)=\mathbf{u}$ $\surd$

1.2 Subspaces

Definition:

Examples:

1) $W = \{(x, y, z) : z = 0\}$. Is W a subspace of $\mathbb{R}^3$?

2) $W = \{(x, y, z) : y = 1\}$ Is W a subspace of $\mathbb{R}^3$?

3) $W = \{(x, y, z) : x^2 + y^2 - z^2 = 0\}$. Is W a subspace of $\mathbb{R}^3$?

1.3 Solution Subspaces

Theorem 1. *If A is a constant $m \times n$ matrix, then the solution set of the system*

$$A\mathbf{x} = \mathbf{0} \tag{1}$$

is a subspace of $\mathbb{R}^n$, called the solution space of the system.

Proof. Let W be the set of all solutions to (1). Verify the conditions for a subspace.

□

1.4 General Vector Spaces

The term "vector" in vector space can be interpreted in a more general sense.

Examples:

(1) Given m and n positive integers, define M_{mn} as the set of all $m \times n$ matrices with real entries. Then M_{mn} is a vector space.

→ matrices play the role of vectors, with

- matrix addition defining "vector" addition
- multiplication of a matrix by a scalar defining scalar multiplication

It is easily verified using these definitions of vector addition and scalar multiplication that M_{mn} satisfies properties (1)-(10) of the definition of a vector space.

(2) $\mathcal{F}$ is the set of all real-valued functions defined on $\mathbb{R}$.

→ functions play the role of vectors.

- For $f, g \in \mathcal{F}$, $(f+g)(x) = f(x) + g(x)$ $\leftarrow$ vector addition
- For $f \in \mathcal{F}$ and $c \in \mathbb{R}$, $(cf)(x) = cf(x)$ $\leftarrow$ scalar multiplication.

It is easily verified using these definitions of vector addition and scalar multiplication that $\mathcal{F}$ is a vector space, with the zero element being the function $f(x) \equiv 0$ (i.e., the function whose value is zero for all x).

(3) $\mathcal{P}$ is the set of all polynomials.

$\mathcal{P}$ is a vector space with the polynomials playing the role of vectors.

Note: $\mathcal{P}$ is also a subspace of $\mathcal{F}$ since $\mathcal{P} \subset \mathcal{F}$.

1.5 Subspaces of General Vector Spaces

Examples:

(1) W is the set of diagonal 2×2 matrices. Is W a subspace of M_{22}?

(2) W is the set of all solutions of the differential equation

$$y' + p(x)y = 0.$$

Is W a subspace of $\mathcal{F}$?

(3) W is the set of all solutions of the differential equation

$$y' + p(x)y = x.$$

Is W a subspace of $\mathcal{F}$?

(4) P_n is the set of all polynomials of degree at most n. Is P_n a subspace of $\mathcal{P}$?

(5) W is the set of all polynomials in P_2 whose coefficients are odd integers. Is W a subspace of P_2?