Quiz #5 Solutions
Math 81, Fall 2009
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1. Find a basis of each eigenspace of the matrix

$$A=\begin{pmatrix}-4 & 2 & 4\\ 0 & 3 & 4\\ 0 & 0 & -4\end{pmatrix}.$$

$$0=\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & 2 & 4\\ 0 & 3-\lambda & 4\\ 0 & 0 & -4-\lambda\end{vmatrix}=(-4-\lambda)(3-\lambda)(-4-\lambda)$$
$$\Rightarrow \lambda=-4\text{ (multiplicity 2)}, 3$$

$\lambda=3$ Find a basis for the solution space of $(A-3I)\vec{v}=\vec{0}$.

$$\begin{pmatrix}-7 & 2 & 4\\ 0 & 0 & 4\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-7 & 2 & 4\\ 0 & 0 & 4\\ 0 & 0 & -7\end{pmatrix}\xrightarrow{r_3\to r_3+\frac{7}{4}r_2}\begin{pmatrix}-7 & 2 & 4\\ 0 & 0 & 4\\ 0 & 0 & 0\end{pmatrix}$$

v_2 is arbitrary
$4v_3=0\Rightarrow v_3=0$
$-7v_1+2v_2+4v_3=0$
$\Rightarrow v_1=2/7\, v_2$
Let $v_2=7r\Rightarrow v_1=2r$

So $\vec{v}=\begin{pmatrix}2r\\7r\\0\end{pmatrix}=r\begin{pmatrix}2\\7\\0\end{pmatrix}$

$\Rightarrow$ A basis of the eigenspace associated with $\lambda=3$ is $\left\{\begin{pmatrix}2\\7\\0\end{pmatrix}\right\}$.

$\lambda=-4$ Find a basis for the solution space of $(A+4I)\vec{v}=\vec{0}$

$$\begin{pmatrix}0 & 2 & 4\\ 0 & 7 & 4\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & 2 & 4\\ 0 & 7 & 4\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{r_2\to r_2-\frac{7}{2}r_1}\begin{pmatrix}0 & 2 & 4\\ 0 & 0 & -10\\ 0 & 0 & 0\end{pmatrix}$$

v_1 is arbitrary
$-10v_3=0\Rightarrow v_3=0$
$2v_2+4v_3=0\Rightarrow v_2=0$

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Let $v_1 = 1 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

A basis of the eigenspace of A corresponding to $\lambda = -4$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

2. Use the eigenvalue method to find a real-valued solution of

$$\vec{x}' = \begin{pmatrix} 3 & 0 & 0 \\ -6 & 5 & 4 \\ 0 & 0 & 3 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

· General solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -6 & 5-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 5-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix}$$
$$= (3-\lambda)(5-\lambda)(3-\lambda)$$
$$\Rightarrow \lambda = 3 \text{ (multiplicity 2)}, \lambda = 5$$

$\boxed{\lambda = 3}$ Solve $(A - 3I)\vec{v} = \vec{0}$ $(\vec{v} \neq \vec{0})$

$$\begin{pmatrix} 0 & 0 & 0 \\ -6 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -6 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -6 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

v_2, v_3 arbitrary

$-6v_1 + 2v_2 + 4v_3 = 0 \rightarrow v_1 = \frac{1}{3}v_2 + \frac{2}{3}v_3$

Let $v_2 = 3r, v_3 = 3s \Rightarrow v_1 = r + 2s$

So $\vec{v} = \begin{pmatrix} r+2s \\ 2r \\ 2s \end{pmatrix} = r\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$\Rightarrow$ Two eigenvectors, $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$\Rightarrow$ Two solutions: $\vec{x}_1 = e^{3t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \vec{x}_2 = e^{3t}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$\boxed{\lambda = 5}$ Next page.

2. $\boxed{\lambda=5}$ Solve $(A-5I)\vec{v}=\vec{0}$ $(\vec{v}\neq\vec{0})$

$$\begin{pmatrix}-2&0&0\\-6&0&4\\0&0&-2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-2&0&0\\-6&0&4\\0&0&-2\end{pmatrix}\xrightarrow{r_2\to r_2-3r_1}\begin{pmatrix}-2&0&0\\0&0&4\\0&0&-2\end{pmatrix}\xrightarrow{r_3\to r_3+\frac{1}{2}r_2}\begin{pmatrix}-2&0&0\\0&0&4\\0&0&0\end{pmatrix}$$

v_2 is arbitrary

$4v_3=0\to v_3=0$

$-2v_1=0\to v_1=0$

Let $v_2=1$

So $\vec{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}\longrightarrow \vec{x}_3=e^{5t}\begin{pmatrix}0\\1\\0\end{pmatrix}$

General solution: $\vec{x}=c_1\vec{x}_1+c_2\vec{x}_2+c_3\vec{x}_3$

$$\vec{x}=c_1e^{3t}\begin{pmatrix}1\\3\\0\end{pmatrix}+c_2e^{3t}\begin{pmatrix}2\\0\\3\end{pmatrix}+c_3e^{5t}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

• Find c_1, c_2, c_3: $\begin{pmatrix}1\\0\\1\end{pmatrix}=\vec{x}(0)=c_1\begin{pmatrix}1\\3\\0\end{pmatrix}+c_2\begin{pmatrix}2\\0\\3\end{pmatrix}+c_3\begin{pmatrix}0\\1\\0\end{pmatrix}$

$c_1+2c_2=1$

$3c_1+c_3=0$

$3c_2=1\to c_2=\frac{1}{3}\Rightarrow c_1=\frac{1}{3}\to c_3=-1$

So

$$\vec{x}(t)=\frac{1}{3}e^{3t}\begin{pmatrix}1\\3\\0\end{pmatrix}+\frac{1}{3}e^{3t}\begin{pmatrix}2\\0\\3\end{pmatrix}-e^{5t}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

3. Use the eigenvalue method to find a real-valued solution of

$$x_1' = -4x_2,\ x_1(0)=5 \qquad \longrightarrow \vec{x}' = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}\vec{x},\ \vec{x}(0) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$
$$x_2' = x_1,\ x_2(0)=2$$

- General solution

$$0 = \det(A-\lambda I) = \begin{vmatrix} -\lambda & -4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+4 \Rightarrow \lambda = \pm 2i$$

$\boxed{\lambda = 2i}$ Solve $(A-2iI)\vec{v} = \vec{0}$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2iv_1 - 4v_2 = 0 \to v_1 = \frac{-2}{i}v_2$

v_2 arbitrary, so let $v_2 = i$

$\Rightarrow v_1 = -2$

So $\vec{v} = \begin{pmatrix} -2 \\ i \end{pmatrix}$

$$e^{\lambda t}\vec{v} = e^{2it}\begin{pmatrix} -2 \\ i \end{pmatrix} = (\cos(2t) + i\sin(2t))\begin{pmatrix} -2 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} -2\cos(2t) - 2i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -2\cos(2t) \\ -\sin(2t) \end{pmatrix}}_{\vec{x}_1} + i\underbrace{\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}}_{\vec{x}_2}$$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 \Rightarrow \vec{x}(t) = c_1\begin{pmatrix} -2\cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

- Find c_1, c_2: $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -2 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$-2c_1 = 5 \Rightarrow c_1 = -5/2$$
$$c_2 = 2$$

$$\vec{x}(t) = \frac{-5}{2}\begin{pmatrix} -2\cos(2t) \\ -\sin(2t) \end{pmatrix} + 2\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos(2t) - 4\sin(2t) \\ \frac{5}{2}\sin(2t) + 2\cos(2t) \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x_1(t) &= 5\cos(2t) - 4\sin(2t) \\ x_2(t) &= \tfrac{5}{2}\sin(2t) + 2\cos(2t) \end{aligned}}$$