Solution to Suggested Problems - Sec 1.6 + Review
Math 81
Instructor: Dr. Doreen De Leon

- Section 1.6: 8, 19, 20, 27, 30, 34, 38
- p. 76: 1, 3, 10, 17, 35 (p 78 in 3rd edition)

- Section 1.6

8. $x^2y' = xy + x^2e^{y/x}$

$y' = y/x + e^{y/x}$

let $v = y/x \to y = xv \Rightarrow y' = xv' + v$

Plug in: $xv' + v = v + e^v$

$xv' = e^v \to x\,dv/dx = e^v$

$\int e^{-v}dv = \int \frac{1}{x}\,dx$

$-e^{-v} = \ln|x| + c$

$\ln(e^{-v} = -\ln|x| + c)$

$-v = \ln|c - \ln|x||$

$v = -\ln|c - \ln|x||$

$\Rightarrow y/x = -\ln|c - \ln|x||$

$\boxed{y = -x\ln|c - \ln|x||}$

19. $x^2y' + 2xy = 5y^3$

$y' + \frac{2}{x}y = \frac{5}{x^2}y^3 \quad (*)$

Bernoulli with $n=3 \Rightarrow$ let $v = y^{1-n} = y^{1-3} \Rightarrow v = y^{-2}$

$v' = -2y^{-3}y'$

From $(*)$, $y' = \frac{5}{x^2}y^3 - \frac{2}{x}y$

$\Rightarrow v' = -2y^{-3}\left(\frac{5}{x^2}y^3 - \frac{2}{x}y\right)$

$= -10/x^2 + \frac{4}{x}y^{-2}$

$= -10/x^2 + 4/x\, v$

$\Rightarrow v' - \frac{4}{x}v = -10/x^2$

$p(x) = -4/x \Rightarrow \rho(x) = e^{\int -4/x\,dx} = e^{-4\ln x} = x^{-4}$

Multiply by $\rho(x)$: $x^{-4}v' - 4x^{-5}v = -10x^{-6}$

$\frac{d}{dx}(x^{-4}v) = -10x^{-6}$

$$\Rightarrow \int \frac{d}{dx}(x^{-4}v)\,dx = \int -10x^{-6}\,dx$$

$$x^{-4}v = 2x^{-5} + c$$

$$v = 2x^{-1} + cx^4$$

$$\boxed{y^{-2} = 2x^{-1} + cx^4}$$

20. $y^2y' + 2xy^3 = 6x$

$y' + 2xy = 6xy^{-2}$ (**)

Bernoulli with $n = -2 \rightarrow$ let $v = y^{1-n} = y^{1-(-2)} \Rightarrow v = y^3$

$v' = 3y^2y'$

From (**), $y' = 6xy^{-2} - 2xy$

$$\Rightarrow v' = 3y^2(6xy^{-2} - 2xy)$$

$$= 18x - 6xy^3$$

$$= 18x - 6xv$$

$v' + 6xv = 18x$

$p(x) = 6x \Rightarrow \rho(x) = e^{\int 6x\,dx} = e^{3x^2}$

Multiply by $\rho(x)$: $e^{3x^2}v' + 6xe^{3x^2}v = 18xe^{3x^2}$

$$\int \frac{d}{dx}(e^{3x^2}v)\,dx = \int 18xe^{3x^2}\,dx$$

$$e^{3x^2}v = 3e^{3x^2} + C$$

$$v = 3 + ce^{-3x^2}$$

$$\Rightarrow y^3 = 3 + ce^{-3x^2}$$

$$\boxed{y = (3 + ce^{-3x^2})^{\frac{1}{3}}}$$

27. $3xy^2y' = 3x^4 + y^3$

$\rightarrow y' = x^3y^{-2} + \frac{1}{3x}y \quad \rightarrow y' - \frac{1}{3x}y = x^3y^{-2}$

Bernoulli with $n = -2$

let $v = y^{1-n} = y^3 \Rightarrow v' = 3y^2y'$

From above, $v' = 3y^2(x^3y^{-2} + \frac{1}{3x}y)$

$$= 3x^3 + \frac{1}{x}y^3 = 3x^3 + \frac{1}{x}v$$

$\Rightarrow v' - \frac{1}{x}v = 3x^3$

$p(x) = \frac{-1}{x} \Rightarrow \rho(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$

Multiply by $\rho(x)$: $x^{-1}v' - x^{-2}v = 3x^2$

$$\int \frac{d}{dx}(x^{-1}v)\,dx = \int 3x^2\,dx \Rightarrow x^{-1}v = x^3 + C$$

$\Rightarrow v = x^4 + cx$

$y^3 = x^4 + cx$

$\boxed{y = (x^4 + cx)^{1/3}}$

70. $dy/dx = y/x - k[1+(y/x)^2]^{1/2}$

let $v = y/x \Rightarrow y = xv \Rightarrow dy/dx = x\frac{dv}{dx} + v$

Plug into the DE: $x\,dv/dx + v = v - k(1+v^2)^{1/2}$

$x\,dv/dx = \cancel{k}\sqrt{1+v^2}$

$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int kx\,dx$

$e^{\ln(v+\sqrt{1+v^2})} = e^{-k\ln|x| + c}$

$v + \sqrt{1+v^2} = cx^{-k}$

$y/x + \sqrt{1+(y/x)^2} = cx^{-k}$

$y + x\sqrt{1+(y/x)^2} = cx^{1-k}$

$y + \sqrt{x^2+y^2} = cx^{1-k}$, $k = a/b = 50/500 = 1/10$

• Find c: $y(200) = 150$

$150 + \sqrt{150^2 + 200^2} = c(200)^{1-1/10}$

$c = 400/200^{9/10} = 2(200)^{1/10}$

So $y + \sqrt{x^2+y^2} = 2(200)^{1/10}\, x^{9/10}$

$\Rightarrow y + \sqrt{x^2+y^2} = 2(200x^9)^{1/10}$ ✓

34. $(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0$

$M(x,y) = 2xy^2 + 3x^2$, $N(x,y) = 2x^2y + 4y^3$

Check if exact: $\partial M/\partial y = 4xy$, $\partial N/\partial x = 4xy = \partial M/\partial y$ ✓

$F(x,y) = \int M(x,y)dx$

$= \int (2xy^2 + 3x^2)dx$

$= x^2y^2 + x^3 + g(y)$

Find g(y): $\partial F/\partial y = 2x^2y + g'(y)$

$\partial F/\partial y = N(x,y) \Rightarrow 2x^2y + g'(y) = 2x^2y + 4y^3$

$g'(y) = 4y^3 \Rightarrow g(y) = \int 4y^3 dy = y^4$

$F(x,y) = x^2y^2 + x^3 + y^4$

Solution: $F(x,y) = c \Rightarrow \boxed{x^2y^2 + x^3 + y^4 = c}$

38. $(x+\tan^{-1}y)dx + \frac{x+y}{1+y^2}dy = 0$

$M(x,y) = x + \tan^{-1}y, \quad N(x,y) = \frac{x+y}{1+y^2}$

Check if exact: $\partial M/\partial y = \frac{1}{1+y^2}, \quad \partial N/\partial x = \frac{1}{1+y^2} = \partial M/\partial y$ ✓

$F(x,y) = \int N(x,y)dy$

$= \int \frac{x+y}{1+y^2} dy$

$= x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) + h(x)$

Find $h(x)$: $\partial F/\partial x = \tan^{-1}y + h'(x)$

$\partial F/\partial x = M(x,y) \Rightarrow \tan^{-1}y + h'(x) = x + \tan^{-1}y$

$h'(x) = x \Rightarrow h(x) = \frac{1}{2}x^2$

So $F(x,y) = x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) + \frac{1}{2}x^2$

General solution: $F(x,y) = c$

$\Rightarrow \boxed{x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) + \frac{1}{2}x^2 = c}$

p. 76: 1, 3, 10, 17, 35

1. $x^3 + 3y - xy' = 0$

$xy' - 3y = x^3$

$y' - \frac{3}{x}y = x^2$ — linear

$P(x) = -3/x \Rightarrow \rho(x) = e^{\int -3/x\, dx} = e^{-3\ln x} = x^{-3}$

Multiply by $\rho(x)$:

$x^{-3}y' - 3/x\,(x^{-3})y = x^2(x^{-3})$

$x^{-3}y' - 3x^{-4}y = x^{-1}$

$\int \frac{d}{dx}(x^{-3}y)dx = \int \frac{1}{x}dx$

$x^{-3}y = \ln|x| + c$

$\boxed{y = x^3\ln|x| + cx^3}$

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3. $xy + y^2 - x^2 y' = 0$

$x^2 y' - xy = y^2$

$y' - \frac{y}{x} = \frac{y^2}{x^2}$ ← homogeneous

$\Rightarrow$ let $v = y/x \rightarrow y = xv$

$\Rightarrow y' = v + xv'$

Plug in: $v + xv' - v = v^2$

$xv' = v^2$

$x \frac{dv}{dx} = v^2 \Rightarrow \int \frac{1}{v^2} dv = \int \frac{1}{x} dx$

$-\frac{1}{v} = \ln|x| + C$

$v = \frac{-1}{\ln|x| + C} \Rightarrow \frac{y}{x} = \frac{-1}{\ln|x| + C}$

So $\boxed{y = \frac{-x}{\ln|x| + C}}$

10. $y' = 1 + x^2 + y^2 + x^2 y^2$

$= 1 + x^2 + y^2(1 + x^2)$

$y' = (1 + y^2)(1 + x^2)$ ← separable

$\frac{dy}{dx} = (1 + y^2)(1 + x^2) \Rightarrow \int \frac{1}{1 + y^2} dy = \int (1 + x^2) dx$

$\tan^{-1} y = x + \frac{1}{3}x^3 + C$

$\boxed{y = \tan(x + \frac{1}{3}x^3 + C)}$

17. $e^x + ye^{xy} + (e^y + xe^{yx})y' = 0$

Check if exact: $M(x,y) = e^x + ye^{xy}$, $N(x,y) = e^y + xe^{yx}$

$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$, $\frac{\partial N}{\partial x} = e^{yx} + xye^{yx} = \frac{\partial M}{\partial y}$ ✓

$F(x,y) = \int M dx$

$= \int (e^x + ye^{xy}) dx$

$= e^x + e^{xy} + g(y)$

$\frac{\partial F}{\partial y} = N(x,y)$ and $\frac{\partial F}{\partial y} = xe^{xy} + g'(y)$

17 (cont.) $\Rightarrow$ $xe^{xy}+g'(y)=e^y+xe^{yx}$

$$g'(y)=e^y \Rightarrow g(y)=e^y$$

So $F(x,y)= e^x+e^{x}y+e^y$

Solution: $F(x,y)=c \Rightarrow \boxed{e^x+e^{x}y+e^y=c}$

35. $\frac{dy}{dx}=\frac{2xy+2x}{x^2+1}$

- Separable: $\frac{dy}{dx}=\frac{2x}{x^2+1}(y+1) \Rightarrow \int\frac{1}{y+1}\,dy=\int\frac{2x}{x^2+1}\,dx$

$$\ln|y+1|=\ln|x^2+1|+c$$

$$y+1=c(x^2+1)$$

$$\boxed{y=c(x^2+1)-1}$$

- Linear: $y'-\frac{2x}{x^2+1}y=\frac{2x}{x^2+1}$

$$p(x)=\frac{-2x}{x^2+1} \Rightarrow \rho(x)=e^{\int\frac{-2x}{x^2+1}dx}=e^{-\ln(x^2+1)}=(x^2+1)^{-1}$$

Multiply by $\rho(x)$: $\frac{1}{x^2+1}y'-\frac{2x}{(x^2+1)^2}y=\frac{2x}{(x^2+1)^2}$

$$\int\frac{d}{dx}\left(\frac{1}{x^2+1}y\right)dx=\int\frac{2x}{(x^2+1)^2}\,dx$$

$$\frac{1}{x^2+1}y=\frac{-1}{x^2+1}+c$$

$$\boxed{y=c(x^2+1)-1}$$

We obtain the same solution with both methods