Solution to Suggested Problems - Sec 10.1 - 10.2
Math 81

- Section 10.1 = 3, 8, 23, 29, 31
- Section 10.2 : 25, 28, 2, 8, 10, 12.

- Section 10.1

3. $f(t) = e^{3t+1}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{3t+1} dt$$
$$= \int_0^\infty e^{-(s-3)t} \cdot e^1 dt$$
$$= e \int_0^\infty e^{-(s-3)t} dt$$
$$= e \cdot \lim_{u\to\infty} \int_0^u e^{-(s-3)t} dt$$
$$= e \cdot \lim_{u\to\infty} \left(\frac{-1}{s-3} e^{-(s-3)t} \right)\Big|_0^u , \quad s \neq 3$$
$$= e \cdot \left(\lim_{u\to\infty} \frac{-1}{s-3} e^{-(s-3)u} + \frac{1}{s-3} \right)$$
$$= \boxed{\frac{e}{s-3}, \quad s > 3}$$

8. $f(t) = \begin{cases} 1, & 1 < t \leq 2 \\ 0, & 0 \leq t \leq 1,\ 2 < t < \infty \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$
$$= \int_0^1 e^{-st} \cdot 0 dt + \int_1^2 e^{-st} \cdot 1 dt + \int_2^\infty e^{-st} \cdot 0 dt$$
$$= -\frac{1}{s} e^{-st}\Big|_1^2 , s > 0 \quad -\frac{1}{s}\left(e^{-2s} - e^{-s}\right)$$
$$= \boxed{\frac{e^{-s} - e^{-2s}}{s}, s > 0}$$

23. $F(s) = \frac{3}{s^4}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \rightarrow 4 = n+1 \Rightarrow n = 3$

$n! = 3! = 6$

(over ⇒)

So $F(s) = \frac{3}{s^4} = \frac{1}{2} \cdot \frac{6}{s^4}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \boxed{\frac{1}{2}t^3}$

29. $F(s) = \frac{5-3s}{s^2+9}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$

$= 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$

$= \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$

So $\boxed{f(t) = \frac{5}{3}\sin(3t) - 3\cos(3t)}$

31. $F(s) = \frac{10s-3}{25-s^2}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 10\mathcal{L}^{-1}\left\{\frac{s}{25-s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{25-s^2}\right\}$

$= -10\mathcal{L}^{-1}\left\{\frac{s}{s^2-25}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-25}\right\}$

$= -10\mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-5^2}\right\}$

$= -10\mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}$

$\rightarrow \boxed{f(t) = -10\cosh(5t) + \frac{3}{5}\sinh(5t)}$

Section 10.2

25. Apply Theorem 1 to derive $\mathcal{L}\{\sin(kt)\}$ from the formula for $\mathcal{L}\{\cos(kt)\}$

Let $f(t) = \sin(kt)$

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Then $f'(t) = K\cos(Kt)$

$$\mathcal{L}\{f'(t)\} = K\mathcal{L}\{\cos(Kt)\} = \frac{Ks}{s^2+K^2}$$

And $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$$\Rightarrow \frac{Ks}{s^2+K^2} = s\mathcal{L}\{f(t)\} - 0 \Rightarrow \mathcal{L}\{f(t)\} = \frac{K}{s^2+K^2} \checkmark$$

28. Show $\mathcal{L}\{t\cos(Kt)\} = \dfrac{s^2-K^2}{(s^2+K^2)^2}$

Let $f(t) = t\cos(Kt)$

$$\Rightarrow f'(t) = \cos(Kt) - Kt\sin(Kt)$$

$$f''(t) = -K\sin(Kt) - K\sin(Kt) - K^2 t\cos(Kt)$$

$$\Rightarrow f''(t) = -2K\sin(Kt) - K^2 f(t) \quad (*)$$

Take the Laplace transform of $(*)$

$$\mathcal{L}\{f''(t)\} = -2K\mathcal{L}\{\sin(Kt)\} - K^2\mathcal{L}\{f(t)\}$$

Since $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$$= s^2\mathcal{L}\{f(t)\} - s(0) - 1$$

$$= s^2\mathcal{L}\{f(t)\} - 1,$$

we obtain

$$s^2\mathcal{L}\{f(t)\} - 1 = -2K\mathcal{L}\{\sin(Kt)\} - K^2\mathcal{L}\{f(t)\}$$

$$(s^2+K^2)\mathcal{L}\{f(t)\} = -2K\mathcal{L}\{\sin(Kt)\} + 1$$

$$= -2K \cdot \frac{K}{s^2+K^2} + 1$$

$$= \frac{s^2-K^2}{s^2+K^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s^2-K^2}{(s^2+K^2)^2}$$

2. $x'' + 9x = 0$, $x(0) = 3$, $x'(0) = 4$

Take the Laplace transform of both sides:

$$\mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 3s - 4$$

$\Rightarrow s^2X(s) - 3s - 4 + 9X(s) = \mathcal{L}\{0\}$

$(s^2+9)X(s) - 3s - 4 = 0$

$(s^2+9)X(s) = 3s+4$

$X(s) = \frac{3s+4}{s^2+9} = \frac{3s}{s^2+9} + \frac{4}{s^2+9}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

$= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

$\Rightarrow \boxed{x(t) = 3\cos(3t) + \frac{4}{3}\sin(3t)}$

8. $x''+9x = 1,\ x(0) = 0 = x'(0)$

Take the Laplace transform of both sides

$\mathcal{L}\{x(t)\} = X(s)$

$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0)$

$= s^2X(s) - s(0) - 0 = s^2X(s)$

$\Rightarrow s^2X(s) + 9X(s) = \mathcal{L}\{1\}$

$(s^2+9)X(s) = 1/s$

$X(s) = \frac{1}{s(s^2+9)}$

Use partial fractions:

$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$

$A = \left.\frac{1}{s^2+9}\right|_{s=0} = \frac{1}{9}$ i.e. (multiply by s, let s=0)

So $\frac{1}{s(s^2+9)} = \frac{1/9}{s} + \frac{Bs+C}{s^2+9}$

Clear fractions:

$1 = 1/9(s^2+9) + (Bs+C)s$

$1 = \frac{1}{9}s^2 + 1 + Bs^2 + Cs$

$-1/9\,s^2 = Bs^2 + Cs$

Equate coefficients: $B = -1/9,\ C = 0$

So $X(s) = \frac{1/9}{s} - \frac{\frac{1}{9}s}{s^2+9}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{9}\mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{9}\mathcal{L}^{-1}\{\frac{s}{s^2+9}\}$

$$\boxed{x(t) = \tfrac{1}{9} - \tfrac{1}{9}\cos(3t)}$$

10. $x'' + 3x' + 2x = t$, $x(0) = 0$, $x'(0) = 2$

Take the Laplace transform of both sides

$\mathcal{L}\{x(t)\} = X(s)$

$\mathcal{L}\{x'(t)\} = sX(s) - s\,x(0)$

$= sX(s)$

$\mathcal{L}\{x''(t)\} = s^2X(s) - s\,x(0) - x'(0)$

$= s^2X(s) - 2$

$\Rightarrow s^2X(s) - 2 + 3sX(s) + 2X(s) = \mathcal{L}\{t\}$

$(s^2+3s+2)X(s) - 2 = \frac{1}{s^2}$

$(s+1)(s+2)X(s) = \frac{1}{s^2} + 2 = \frac{1+2s^2}{s^2}$

$X(s) = \frac{2s^2+1}{s^2(s+1)(s+2)}$

Use partial fractions

$\frac{2s^2+1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$

$B = \left.\frac{2s^2+1}{(s+1)(s+2)}\right|_{s=0} = \frac{1}{1\cdot 2} = \frac{1}{2}$

$C = \left.\frac{2s^2+1}{s^2(s+2)}\right|_{s=-1} = \frac{2(-1)^2+1}{(-1)^2(-1+2)} = 3$

$D = \left.\frac{2s^2+1}{s^2(s+1)}\right|_{s=-2} = \frac{2(-2)^2+1}{(-2)^2(-2+1)} = \frac{-9}{4}$

So $\frac{2s^2+1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{1/2}{s^2} + \frac{3}{s+1} - \frac{9/4}{s+2}$

Clear fractions:

$2s^2+1 = As(s+1)(s+2) + \frac{1}{2}(s+1)(s+2) + 3s^2(s+2) - \frac{9}{4}s^2(s+1)$

$2s^2+1 = A(s^3+3s^2+2s) + \frac{1}{2}(s^2+3s+2) + 3s^3+6s^2 - \frac{9}{4}s^3 - \frac{9}{4}s^2$

$2s^2+1 = As^3 + 3As^2 + 2As + \frac{3}{4}s^3 + \frac{17}{4}s^2 + \frac{3}{2}s + 1$

$\Rightarrow -\frac{3}{4}s^3 - \frac{9}{4}s^2 - \frac{3}{2}s = As^3 + 3As^2 + 2As$

Equate coefficients: $A = -\frac{3}{4}$

So $X(s) = \frac{-3/4}{s} + \frac{1/2}{s^2} + \frac{3}{s+1} - \frac{9/4}{s+2}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = -\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$$\boxed{x(t) = -\frac{3}{4} + \frac{1}{2}t + 3e^{-t} - \frac{9}{4}e^{-2t}}$$

12. $x' = x + 2y$, $x(0) = 0$

$y' = x + e^{-t}$, $y(0) = 0$

Take the Laplace transform of both equations

$sX(s) - x(0) = X(s) + 2Y(s) \Rightarrow sX(s) = X(s) + 2Y(s)$ (a)

$sY(s) - y(0) = X(s) + \frac{1}{s+1} \Rightarrow sY(s) = X(s) + \frac{1}{s+1}$ (b)

(a) $\rightarrow \frac{(s-1)}{2}X(s) = Y(s)$

Plug into (b): $s\frac{(s-1)}{2}X(s) = X(s) + \frac{1}{s+1}$

$$\left(\frac{s(s-1)}{2} - 1\right)X(s) = \frac{1}{s+1}$$

$$\left(\frac{1}{2}s^2 - \frac{1}{2}s - 1\right)X(s) = \frac{1}{s+1}$$

$$(s^2 - s - 2)X(s) = \frac{2}{s+1}$$

$$(s-2)(s+1)X(s) = \frac{2}{s+1}$$

$$X(s) = \frac{2}{(s-2)(s+1)^2}$$

Use partial fractions:

$$\frac{2}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = \frac{2}{(s+1)^2}\Big|_{s=2} = \frac{2}{(2+1)^2} = \frac{2}{9}$$

$$C = \frac{2}{s-2}\Big|_{s=-1} = \frac{2}{-1-2} = \frac{-2}{3}$$

So $\frac{2}{(s-2)(s+1)^2} = \frac{2/9}{s-2} + \frac{B}{s+1} - \frac{2/3}{(s+1)^2}$

Clear fractions:

$$2 = 2/9(s+1)^2 + B(s-2)(s+1) - 2/3(s-2)$$

$$2 = 2/9(s^2+2s+1) + B(s^2-s-2) - 2/3 s + 4/3$$

$$2 = 2/9 s^2 + 4/9 s + 2/9 + Bs^2 - Bs - 2B - 2/3 s + 4/3$$

$$2 = 2/9 s^2 - 2/9 s + 14/9 + Bs^2 - Bs - 2B$$

$$-2/9 s^2 + 2/9 s + 4/9 = Bs^2 - Bs - 2B$$

Equate coefficients: $B = -2/9$

$$\Rightarrow X(s) = \frac{2/9}{s-2} - \frac{2/9}{s+1} - \frac{2/3}{(s+1)^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2/9\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$\boxed{x(t) = 2/9\, e^{2t} - 2/9\, e^{-t} - 2/3\, te^{-t}}$$

from Ex. 4, p. 584
(we will do this another way later)

Find $y(t)$: $Y(s) = \frac{(s-1)}{2}X(s) = \frac{(s-1)}{(s-2)(s+1)^2}$

Use partial fractions

$$\frac{s-1}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = \frac{s-1}{(s+1)^2}\Big|_{s=2} = \frac{2-1}{(2+1)^2} = \frac{1}{9}$$

$$C = \frac{s-1}{s-2}\Big|_{s=-1} = \frac{-1-1}{-1-2} = \frac{2}{3}$$

So $\frac{s-1}{(s-2)(s+1)^2} = \frac{1/9}{s-2} + \frac{B}{s+1} + \frac{2/3}{(s+1)^2}$

Clear fractions:

$$s-1 = 1/9(s+1)^2 + B(s-2)(s+1) + 2/3(s-2)$$

$$s-1=\tfrac{1}{9}(s^2+2s+1)+B(s^2-s-2)+\tfrac{2}{3}s-\tfrac{4}{3}$$

$$s-1=\tfrac{1}{9}s^2+\tfrac{2}{9}s+\tfrac{1}{9}+Bs^2-Bs-2B+\tfrac{2}{3}s-\tfrac{4}{3}$$

$$-\tfrac{1}{9}s^2+\tfrac{1}{9}s+\tfrac{2}{9}=Bs^2-Bs-2B$$

Equate coefficients: $B=-\tfrac{1}{9}$

So $Y(s)=\dfrac{1/9}{s-2}-\dfrac{1/9}{s+1}+\dfrac{2/3}{(s+1)^2}$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$\boxed{y(t)=\tfrac{1}{9}e^{2t}-\tfrac{1}{9}e^{-t}+\tfrac{2}{3}te^{-t}}$$

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