Solutions to Suggested Problems
Math 87
Instructor: Dr. Doreen De Leon

- Section 10.3: 1, 3, 7, 9, 10, 27, 30

1. Find $\mathcal{L}\{t^4 e^{\pi t}\}$

$$\mathcal{L}\{t^4\} = \frac{4!}{s^{4+1}} = \frac{24}{s^5} = F(s) \rightarrow \mathcal{L}\{t^4 e^{\pi t}\} = F(s-\pi) = \boxed{\frac{24}{(s-\pi)^5}}$$

3. $\mathcal{L}\{e^{-2t}\sin(3\pi t)\}$

$$\mathcal{L}\{\sin(3\pi t)\} = \frac{3\pi}{s^2+(3\pi)^2} = F(s) \longrightarrow \mathcal{L}\{e^{-2t}\sin(3\pi t)\} = F(s+2) = \boxed{\frac{3\pi}{(s+2)^2+9\pi^2}}$$

7. $F(s) = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \boxed{te^{-2t}}$$

9. $F(s) = \frac{3s+5}{s^2-6s+25}$

$$\frac{3s+5}{s^2-6s+25} = \frac{3s+5}{s^2-6s+9-9+25} = \frac{3s+5}{(s-3)^2+16}$$

$$= \frac{3(s-3)+9+5}{(s-3)^2+16}$$

$$= \frac{3(s-3)+14}{(s-3)^2+16} = \frac{3(s-3)}{(s-3)^2+16} + \frac{14}{(s-3)^2+16}$$

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+16}\right\} + 14\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+16}\right\}$$

$$= 3e^{3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + 14\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$$

$$= 3e^{3t}\cos(4t) + \frac{14}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \boxed{3e^{3t}\cos(4t) + \frac{7}{2}e^{3t}\sin(4t)}$$

10. $F(s)=\dfrac{2s-3}{9s^2-12s+20}$

$$\frac{2s-3}{9s^2-12s+20}=\frac{2s-3}{9(s^2-\frac{4}{3}s)+20}=\frac{2s-3}{9(s^2-\frac{4}{3}s+\frac{4}{9}-\frac{4}{9})+20}$$

$$=\frac{2s-3}{9(s^2-\frac{4}{3}s+\frac{4}{9})-4+20}$$

$$=\frac{2s-3}{9(s-\frac{2}{3})^2+16}$$

$$=\frac{2(s-\frac{2}{3}+\frac{2}{3})-3}{9(s-\frac{2}{3})^2+16}=\frac{2(s-\frac{2}{3})+\frac{4}{3}-3}{9(s-\frac{2}{3})^2+16}$$

$$=\frac{2(s-\frac{2}{3})}{9(s-\frac{2}{3})^2+16}-\frac{5/3}{9(s-\frac{2}{3})^2+16}$$

$$=\frac{2/9\,(s-\frac{2}{3})}{(s-\frac{2}{3})^2+16/9}-\frac{5/27}{(s-\frac{2}{3})^2+16/9}$$

$$\mathcal{L}^{-1}\{F(s)\}=\frac{2}{9}\mathcal{L}^{-1}\left\{\frac{s-2/3}{(s-\frac{2}{3})^2+(16/9)}\right\}-\frac{5}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s-\frac{2}{3})^2+(16/9)}\right\}$$

$$=\frac{2}{9}e^{2/3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\frac{4}{3})^2}\right\}-\frac{5}{27}e^{2/3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\frac{4}{3})^2}\right\}$$

$$=\frac{2}{9}e^{2/3t}\cos(\tfrac{4}{3}t)-\frac{5}{27}\cdot\frac{3}{4}e^{2/3t}\mathcal{L}^{-1}\left\{\frac{4/3}{s^2+(\frac{4}{3})^2}\right\}$$

$$=\boxed{\frac{2}{9}e^{2/3t}\cos(\tfrac{4}{3}t)-\frac{5}{36}e^{2/3t}\sin(\tfrac{4}{3}t)}$$

27. $x''+6x'+25x=0,\ x(0)=2,\ x'(0)=3$

Take Laplace transform of both sides

$$\mathcal{L}\{x''\}+6\mathcal{L}\{x'\}+25\mathcal{L}\{x\}=\mathcal{L}\{0\}$$

$$\mathcal{L}\{x'\}=sX(s)-x(0)=sX(s)-2$$

$$\mathcal{L}\{x''\}=s^2X(s)-sx(0)-x'(0)=s^2X(s)-2s-3$$

So $s^2X(s)-2s-3+6(sX(s)-2)+25X(s)=0$

$$s^2X(s)-2s-3+6sX(s)-12+25X(s)=0$$

$$(s^2+6s+25)X(s)=2s+15$$

$$X(s)=\frac{2s+15}{s^2+6s+25}=\frac{2s+15}{s^2+6s+9-9+25}$$

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$$X(s) = \frac{2s+15}{(s+3)^2+16}$$

$$= \frac{2(s+3-3)+15}{(s+3)^2+16} = \frac{2(s+3)-6+15}{(s+3)^2+16}$$

$$= \frac{2(s+3)}{(s+3)^2+16} + \frac{9}{(s+3)^2+16}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+16}\right\} + 9\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+16}\right\}$$

$$= 2e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \frac{9}{4}e^{-3t}\mathcal{L}^{-1}\left\{\frac{1\cdot 4}{s^2+16}\right\}$$

$$\boxed{x(t) = 2e^{-3t}\cos(4t) + \tfrac{9}{4}e^{-3t}\sin(4t)}$$

30. $x'' + 4x' + 8x = e^{-t}$, $x(0) = x'(0) = 0$

Take the Laplace transform of both sides

$\mathcal{L}\{x'\} = sX(s) - x(0) = sX(s)$

$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0) = s^2X(s)$

$\rightarrow s^2X(s) + 4sX(s) + 8X(s) = \mathcal{L}\{e^{-t}\}$

$(s^2+4s+8)X(s) = \frac{1}{s+1}$

$$X(s) = \frac{1}{(s+1)(s^2+4s+8)}$$

$$\frac{1}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8}$$

A) $A = \left.\frac{1}{s^2+4s+8}\right|_{s=-1} = \frac{1}{5}$

$$\frac{1}{(s+1)(s^2+4s+8)} = \frac{1/5}{s+1} + \frac{Bs+C}{s^2+4s+8}$$

Clear fractions: $1 = \frac{1}{5}(s^2+4s+8) + (Bs+C)(s+1)$

$1 = \frac{1}{5}s^2 + \frac{4}{5}s + \frac{8}{5} + Bs^2 + Cs + Bs + C$

$1 = \frac{1}{5}s^2 + \frac{4}{5}s + \frac{8}{5} + Bs^2 + (B+C)s + C$

$-\frac{1}{5}s^2 - \frac{4}{5}s - \frac{3}{5} = Bs^2 + (B+C)s + C$

Equate coefficients: $B = -\frac{1}{5}$, $C = -\frac{3}{5}$

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So $X(s) = \frac{1/5}{s+1} + \frac{-\frac{1}{5}s - 3/5}{s^2+4s+8} = \frac{1/5}{s+1} + \frac{-\frac{1}{5}s - 3/5}{(s+2)^2+4}$

$= \frac{1/5}{s+1} + \frac{-\frac{1}{5}(s+2-2) - 3/5}{(s+2)^2+4} = \frac{1/5}{s+1} + \frac{-\frac{1}{5}(s+2) - \frac{1}{5}}{(s+2)^2+4}$

$= \frac{1/5}{s+1} - \frac{1/5(s+2)}{(s+2)^2+4} - \frac{1/5}{(s+2)^2+4}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}$

$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{5}e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$\boxed{x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos(2t) - \frac{1}{10}e^{-2t}\sin(2t)}$$

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