Solutions to Suggested Problems - Secs 10.4-10.5
Math 81
Instructor: Dr. Doreen De Leon

Section 10.4: 3, 12
Section 10.5: 6, 8, 10, 11, 14

Section 10.4

3. $f(t) = \sin t,\ g(t) = \sin t$

$$f * g = \int_0^t \sin\tau \cdot \sin(t-\tau)\,d\tau$$
$$= \int_0^t \frac{1}{2}\left(\cos(\tau-(t-\tau)) - \cos(\tau+(t-\tau))\right)d\tau$$
$$= \frac{1}{2}\int_0^t \cos(2\tau - t)\,d\tau - \frac{1}{2}\int_0^t \cos t\,d\tau$$
$$= \frac{1}{4}\sin(2\tau - t)\Big|_0^t - \frac{1}{2}\cos t\cdot\tau\Big|_0^t = \frac{1}{4}\sin t - \frac{1}{4}\sin(-t) - \frac{1}{2}t\cos t$$
$$= \boxed{\frac{1}{2}\sin t - \frac{1}{2}t\cos t}$$

12. $$F(s) = \frac{1}{s(s^2+4s+5)} = \frac{1}{s}\cdot\frac{1}{s^2+4s+5}$$
$$= \frac{1}{s}\cdot\frac{1}{s^2+4s+\underline{4}-\underline{4}+5}$$
$$= \underbrace{\frac{1}{s}}_{G(s)}\cdot\underbrace{\frac{1}{(s+2)^2+1}}_{H(s)}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$
$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-2t}\sin t$$
$$\mathcal{L}^{-1}\{F(s)\} = g*h = \int_0^t 1\cdot e^{-2\tau}\sin\tau\,d\tau$$
$$= \int_0^t e^{-2\tau}\sin\tau\,d\tau$$
$$= \boxed{\frac{1}{5} - \frac{1}{5}e^{-2t}\cos t - \frac{2}{5}e^{-2t}\sin t}$$

Section 10.5

6. $F(s) = \frac{se^{-s}}{s^2+\pi^2}$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos(\pi t) \Rightarrow \mathcal{L}^{-1}\left\{\frac{se^{-s}}{s^2+\pi^2}\right\} = \boxed{\cos(\pi(t-1))u(t-1)}$

8. $F(s) = \frac{s(1-e^{-2s})}{s^2+\pi^2} = \frac{s}{s^2+\pi^2} - e^{-2s}\cdot\frac{s}{s^2+\pi^2}$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos(\pi t)$

$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{s}{s^2+\pi^2}\right\} = \cos(\pi(t-2))u(t-2) = \cos(\pi t - 2\pi)u(t-2)$
$= \cos(\pi t)u(t-2)$

So $\mathcal{L}^{-1}\{F(s)\} = \cos(\pi t) - \cos(\pi t)u(t-2)$
$= \cos(\pi t)(1-u(t-2))$

10. $F(s) = \frac{2s(e^{-\pi s}-e^{-2\pi s})}{s^2+4} = \frac{2s}{s^2+4}\cdot e^{-\pi s} - \frac{2s}{s^2+4}e^{-2\pi s}$

$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = 2\cos(2t)$

$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}e^{-\pi s}\right\} = 2\cos(2(t-\pi))u(t-\pi)$
$= 2\cos(2t-2\pi)u(t-\pi) = 2\cos(2t)u(t-\pi)$

$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\cdot e^{-2\pi s}\right\} = 2\cos(2(t-2\pi))u(t-2\pi)$
$= 2\cos(2t-4\pi)u(t-2\pi) = 2\cos(2t)u(t-2\pi)$

So, $\mathcal{L}^{-1}\{F(s)\} = 2\cos(2t)u(t-\pi) - 2\cos(2t)u(t-2\pi)$
$= 2\cos(2t)[u(t-\pi) - u(t-2\pi)]$

or $f(t) = \begin{cases} 2\cos(2t), & \pi \le t < 2\pi \\ 0, & \text{otherwise} \end{cases}$

11. $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases} \Rightarrow f(t) = 2(u(t) - u(t-3))$
$= 2u(t) - 2u(t-3)$
$= 2 - 2u(t-3)$

11 (cont.) So $\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} - \mathcal{L}\{2u(t-3)\}$

$$= \frac{2}{s} - 2\mathcal{L}\{u(t-3)\}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-3s} = \boxed{\frac{2}{s}\left(1-e^{-3s}\right)}$$

14. $f(t) = \begin{cases} \cos(\pi t), & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow f(t) = \cos(\pi t)(u(t) - u(t-2))$

$$= \cos(\pi t)\,u(t) - \cos(\pi t)\,u(t-2)$$

$$= \cos(\pi t) - \cos(\pi t)\,u(t-2)$$

$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$

$\mathcal{L}\{\cos(\pi t)u(t-2)\} = ?$

Need $\cos(\pi(t-2))\,u(t-2)$:

$\cos(\pi(t-2))\,u(t-2) = \cos(\pi t - 2\pi)u(t-2) = \cos(\pi t)u(t-2)$

So, $\cos(\pi(t-2))u(t-2) = \cos(\pi t)\,u(t-2)$

$\Rightarrow \mathcal{L}\{\cos(\pi t)\,u(t-2)\} = \mathcal{L}\{\cos(\pi(t-2))u(t-2)\}$

$$= e^{-2s}\mathcal{L}\{\cos(\pi t)\}$$

$$= e^{-2s}\cdot\frac{s}{s^2+\pi^2}$$

So $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(\pi t)\} - \mathcal{L}\{\cos(\pi t)u(t-2)\}$

$$= \frac{s}{s^2+\pi^2} - \frac{e^{-2s}\cdot s}{s^2+\pi^2}$$

$$= \boxed{\frac{s}{s^2+\pi^2}\left(1-e^{-2s}\right)}$$