Solutions to Suggested Problems - Sec 4.1-4.2, 4.7
Math 81
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- Section 4.1: 30, 31, 34, 35
- Section 4.2: 5, 6, 7, 10, 14, 15, 20
- Section 4.7: 1, 6, 7, 9, 10, 11

Section 4.1

30. $V = \{(x, y, z) : x + y + z = 0\}$

$\vec{0} = (0,0,0) \in V$ since $0+0+0=0$

$\vec{0} \in V \Rightarrow V$ could be a subspace of $\mathbb{R}^3$

- closure under vector addition

Let $\vec{u} = (u_1, u_2, u_3) \in V$ (so $u_1 + u_2 + u_3 = 0$)

$\vec{v} = (v_1, v_2, v_3) \in V$ (so $v_1 + v_2 + v_3 = 0$)

Need to show that $\vec{u} + \vec{v} \in V$

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$\vec{u} + \vec{v} \in$ if $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = 0$

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3)$$
$$= (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$$
$$= 0 + 0 = 0 \checkmark$$

So $\vec{u} + \vec{v} \in V$

- closure under scalar multiplication

Let c be a scalar

$c\vec{u} = (cu_1, cu_2, cu_3)$

Need to show that $c\vec{u} \in V$

$\Rightarrow$ Need to verify that $(cu_1) + (cu_2) + (cu_3) = 0$

$$(cu_1) + (cu_2) + (cu_3) = c(u_1 + u_2 + u_3)$$
$$= c(0)$$
$$= 0 \checkmark$$

So $c\vec{u} \in V$

Since $V \subset \mathbb{R}^3$ is closed under vector addition and scalar multiplication, V is a subspace of $\mathbb{R}^3$.

31. $V=\{(x,y,z): 2x=3y\}$

$\vec{0}=(0,0,0)\in V$ since $2(0)=3(0)$

Since $\vec{0}\in V$, V could be a subspace of $\mathbb{R}^3$

• Closure under vector addition

Let $\vec{u}=(u_1,u_2,u_3)\in V$ (so $2u_1=3u_2$)

$\vec{v}=(v_1,v_2,v_3)\in V$ (so $2v_1=3v_2$)

Need to verify $\vec{u}+\vec{v}\in V$

$\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2, u_3+v_3)$

$\vec{u}+\vec{v}\in V$ if $2(u_1+v_1)=3(u_2+v_2)$

$$\begin{aligned}2(u_1+v_1) &= 2u_1+2v_1\\ &= 3u_2+3v_2\\ &= 3(u_2+v_2)\ \checkmark\end{aligned}$$

So $\vec{u}+\vec{v}\in V$ ✓

• Closure under scalar multiplication

Let c be a scalar

Need to check if $c\vec{u}\in V$

$c\vec{u}=(cu_1, cu_2, cu_3)$

$c\vec{u}\in V$ if $2(cu_1)=3(cu_2)$

$$\begin{aligned}2(cu_1) &= c(2u_1)\\ &= c(3u_2)\\ &= 3(cu_2)\ \checkmark\end{aligned}$$

So $c\vec{u}\in V$ ✓

Since $V\subset\mathbb{R}^3$ is closed under vector addition and scalar multiplication, V is a subspace of $\mathbb{R}^3$.

34. $V=\{(x,y,z): x+y+z=3\}$

$\vec{0}=(0,0,0)$

$\vec{0}\notin V$ because $0+0+0\neq 3$.

$\vec{0}\notin V \Rightarrow V$ is not a subspace of $\mathbb{R}^3$.

35. $V=\{(x,y,z): z\geq 0\}$

$\vec{0}=(0,0,0)\in V$

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But let $\vec{u} = (0,0,1) \in V$ and $c = -1$

Then $c\vec{u} = (0,0,-1) \notin V$

So V is not closed under scalar multiplication

$\Rightarrow V$ is not a subspace of $\mathbb{R}^3$.

Section 4.2

5. W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

Is W a subspace of $\mathbb{R}^4$?

$\vec{0} = (0,0,0,0) \in W$ since $0 + 2(0) + 3(0) + 4(0) = 0$

Therefore W could be a subspace of $\mathbb{R}^4$.

- closure under vector addition

Let $\vec{u} = (u_1, u_2, u_3, u_4) \in W$ (so $u_1 + 2u_2 + 3u_3 + 4u_4 = 0$)

$\vec{v} = (v_1, v_2, v_3, v_4) \in W$ (so $v_1 + 2v_2 + 3v_3 + 4v_4 = 0$)

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)$

Need to verify that $\vec{u} + \vec{v} \in W$

$\vec{u} + \vec{v} \in W$ if $(u_1+v_1) + 2(u_2+v_2) + 3(u_3+v_3) + 4(u_4+v_4) = 0$

$$(u_1+v_1) + 2(u_2+v_2) + 3(u_3+v_3) + 4(u_4+v_4)$$
$$= (u_1 + 2u_2 + 3u_3 + 4u_4) + (v_1 + 2v_2 + 3v_3 + 4v_4)$$
$$= 0 + 0 = 0 \checkmark$$

So $\vec{u} + \vec{v} \in W \checkmark$

- closure under scalar multiplication

Let c be a scalar

$c\vec{u} = (cu_1, cu_2, cu_3, cu_4)$

Need to verify that $c\vec{u} \in W$

$c\vec{u} \in W$ if $(cu_1) + 2(cu_2) + 3(cu_3) + 4(cu_4) = 0$

$$(cu_1) + 2(cu_2) + 3(cu_3) + 4(cu_4) = c(u_1 + 2u_2 + 3u_3 + 4u_4)$$
$$= c(0)$$
$$= 0 \checkmark$$

So $c\vec{u} \in W \checkmark$

Since $W \subset \mathbb{R}^4$ is closed under vector addition and scalar multiplication, W is a subspace of $\mathbb{R}^4$.

6. W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$.

$\vec{0} = (0,0,0,0) \in W$ since $0 = 3(0)$ and $0 = 4(0)$.

$\vec{0} \in W \Rightarrow W$ could be a subspace of $\mathbb{R}^4$.

- closure under vector addition

Let $\vec{u} = (3u_3, 4u_4, u_3, u_4) \in W$

$\vec{v} = (3v_3, 4v_4, v_3, v_4) \in W$

Need to verify $\vec{u} + \vec{v} \in W$

$$\vec{u} + \vec{v} = (3u_3 + 3v_3, 4u_4 + 4v_4, u_3 + v_3, u_4 + v_4)$$
$$= (3(u_3+v_3), 4(u_4+v_4), u_3+v_3, u_4+v_4) \in W \checkmark$$

- closure under scalar multiplication

Let c be a scalar.

Need to verify $c\vec{u} \in W$

$$c\vec{u} = c(3u_3, 4u_4, u_3, u_4)$$
$$= (3(cu_3), 4(cu_4), cu_3, cu_4) \in W \checkmark$$

Since $W \subset \mathbb{R}^4$ is closed under vector addition and scalar multiplication, W is a subspace of $\mathbb{R}^4$.

7. W is the set of all vectors in $\mathbb{R}^2$ such that $|x_1| = |x_2|$.

$\vec{0} = (0,0) \in W$ since $|0| = |0|$

So W could be a subspace of $\mathbb{R}^2$.

But, if we let $\vec{u} = (1,-1) \in W$ since $|1| = |-1|$

$\vec{v} = (1,1) \in W$ since $|1| = |1|$

$$\vec{u} + \vec{v} = (2,0) \notin W \text{ since } |2| \neq |0|$$

Since W is not closed under vector addition, W is not a subspace of $\mathbb{R}^2$.

10. W is the set of all vectors in $\mathbb{R}^2$ such that $|x_1| + |x_2| = 1$.

$\vec{0} = (0,0) \notin W$ since $|0| + |0| \neq 1$

$\vec{0} \notin W \Rightarrow W$ is not a subspace of $\mathbb{R}^2$.

14. W is the set of all vectors in $\mathbb{R}^4$ whose components are all nonzero.

$\vec{0} = (0,0,0,0) \notin W \Rightarrow W$ is not a subspace of $\mathbb{R}^4$.

15.
$$x_1 - 4x_2 + x_3 - 4x_4 = 0$$
$$x_1 + 2x_2 + x_3 + 8x_4 = 0$$
$$x_1 + x_2 + x_3 + 6x_4 = 0$$

Solve by Gaussian elimination

$$\begin{pmatrix} 1 & -4 & 1 & -4 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 6 \end{pmatrix} \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & -4 & 1 & -4 \\ 0 & 6 & 0 & 12 \\ 0 & 5 & 0 & 10 \end{pmatrix} \xrightarrow{r_2 \to \frac{1}{6} r_2}$$

$$\begin{pmatrix} 1 & -4 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 0 & 10 \end{pmatrix} \xrightarrow{r_3 \to r_3 - 5r_2} \begin{pmatrix} 1 & -4 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3, x_4 are arbitrary, so let $x_3 = s,\ x_4 = t$

$$x_2 + 2x_4 = 0 \Rightarrow x_2 = -2x_4 = -2t$$
$$x_1 - 4x_2 + x_3 - 4x_4 = 0$$
$$x_1 = 4x_2 - x_3 + 4x_4$$
$$= 4(-2t) - s + 4t$$
$$= -4t - s$$

so
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s-4t \\ -2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -4t \\ -2t \\ 0 \\ t \end{pmatrix}$$
$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

20.
$$\begin{matrix} x_1 + 5x_2 + x_3 - 8x_4 = 0 \\ 2x_1 + 5x_2 \quad - 5x_4 = 0 \\ 2x_1 + 7x_2 + x_3 - 9x_4 = 0 \end{matrix} \rightarrow \begin{pmatrix} 1 & 5 & 1 & -8 \\ 2 & 5 & 0 & -5 \\ 2 & 7 & 1 & -9 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - 2r_1}$$

$$\begin{pmatrix} 1 & 5 & 1 & -8 \\ 0 & -5 & -2 & 11 \\ 0 & -3 & -1 & 7 \end{pmatrix} \xrightarrow{r_3 \to r_3 - \frac{3}{5} r_2} \begin{pmatrix} 1 & 5 & 1 & -8 \\ 0 & -5 & -2 & 11 \\ 0 & 0 & 1/5 & 2/5 \end{pmatrix}$$

x_4 arbitrary $\Rightarrow$ Let $x_4 = s$

$\frac{1}{5} x_3 + \frac{2}{5} x_4 = 0 \Rightarrow x_3 = -2x_4 = -2s$

$-5x_2 - 2x_3 + 11x_4 = 0 \Rightarrow x_2 = -\frac{2}{5} x_3 + \frac{11}{5} x_4$

$= -\frac{2}{5}(-2s) + \frac{11}{5} s$

$= 3s$

$x_1 + 5x_2 + x_3 - 8x_4 = 0 \Rightarrow x_1 = -5x_2 - x_3 + 8x_4$

$= -5(3 \cdot s) + 2s + 8s$

$= -5s$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5s \\ 3s \\ -2s \\ s \end{pmatrix} = s \begin{pmatrix} -5 \\ 3 \\ -2 \\ 1 \end{pmatrix}$

Section 4.7

1. W = the set of all diagonal 3×3 matrices

The zero vector for 3×3 matrices is the zero matrix,

$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W$ (it is diagonal)

So W could be a subspace of M_{33}

- Closure under addition

Let $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in W$

$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_1 \end{pmatrix} \in W$

$A + B = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & a_2+b_2 & 0 \\ 0 & 0 & a_3+b_3 \end{pmatrix} \in W$ ✓

- closure under scalar multiplication

let c be a scalar

$$cA = \begin{pmatrix} ca_1 & 0 & 0 \\ 0 & ca_2 & 0 \\ 0 & 0 & ca_3 \end{pmatrix} \in W \checkmark$$

Since $W \subset M_{33}$ is closed under addition and scalar multiplication, W is a subspace of M_{33}.

6. W = the set of all f such that $f(x) \neq 0$ for all x

The zero element for $\mathcal{F}$ is the zero function, $z(x) \equiv 0$. The zero function is not in W. Therefore W is not a subspace of $\mathcal{F}$.

7. W = the set of all f such that $f(0)=0$ and $f(1)=1$.

Again, the zero element for $\mathcal{F}$ is the zero function, $z(x) \equiv 0. \Rightarrow z(1) = 0 \neq 1$.

So, $z \notin W \Rightarrow W$ is not a subspace of $\mathcal{F}$.

9. W = the set of polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that $a_3 \neq 0$.

The zero vector for $\mathcal{P}$ is the zero polynomial, $z(x) \equiv 0$.

$z \notin W$ since $a_3 \neq 0 \Rightarrow W$ is not a subspace of $\mathcal{P}$.

10. W = the set of polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that $a_0 = a_1 = 0$.

The zero element for P_3 is the zero polynomial, $z_3(x) = 0 + 0x + 0x^2 + 0x^3 \Rightarrow z_3 \in W$, so W could be a subspace of P_3.

- closure under addition

let $p \in W \Rightarrow p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, $a_0 = a_1 = 0$

$\Rightarrow p(x) = a_2 x^2 + a_3 x^3$

$q \in W \Rightarrow q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$, $b_0 = b_1 = 0$

$\Rightarrow q(x) = b_2 x^2 + b_3 x^3$

Need to verify $p+q \in W$

$(p+q)(x) = p(x)+q(x) = (a_2+b_2)x^2 + (a_3+b_3)x^3 \in W$ ✓

- Closure under scalar multiplication

Let c be a scalar.

Need to verify $cp \in W$

$(cp)(x) = cp(x) = ca_2x^2 + ca_3x^3 \in W$ ✓

Since $W \subset \mathbb{P}$ is closed under addition and scalar multiplication, W is a subspace of $\mathbb{P}$.

11. W = the set of all polynomials of the form $a_0+a_1x+a_2x^2+a_3x^3$ such that $a_0+a_1+a_2+a_3=0$

The zero polynomial, $z_3(x) = 0+0x+0x^2+0x^3$ is in W $(0+0+0+0=0)$

So W could be a subspace of $\mathbb{P}$.

- Closure under addition

Let $p \in W \Rightarrow p(x) = a_0+a_1x+a_2x^2+a_3x^3$, with $a_0+a_1+a_2+a_3=0$

$q \in W \Rightarrow q(x) = b_0+b_1x+b_2x^2+b_3x^3$, with $b_0+b_1+b_2+b_3=0$

Need to verify $p+q \in W$

$(p+q)(x) = p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+(a_3+b_3)x^3$

$p+q \in W$ if $(a_0+b_0)+(a_1+b_1)+(a_2+b_2)+(a_3+b_3)=0$

$(a_0+b_0)+(a_1+b_1)+(a_2+b_2)+(a_3+b_3)$

$= (a_0+a_1+a_2+a_3)+(b_0+b_1+b_2+b_3)$

$= 0+0 = 0$ ✓

So $p+q \in W$ ✓

- Closure under scalar multiplication

Let c be a scalar

Need to verify $cp \in W$

$(cp)(x) = c \cdot p(x) = ca_0 + ca_1x + ca_2x^2 + ca_3x^2$

$cp \in W$ if $(ca_0)+(ca_1)+(ca_2)+(ca_3)=0$

$(ca_0)+(ca_1)+(ca_2)+(ca_3) = c(a_0+a_1+a_2+a_3)$

$= c(0) = 0$ ✓

So $cp \in W$

Since $W \subset \mathbb{P}$ is closed under addition and scalar multiplication W is a subspace of $\mathbb{P}$.

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