Solutions to Suggested Problems
Math 81
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- Section 5.1: 3, 16, 20, 25
- Section 5.2: 21, 7, 16
- Section 5.3: 2, 4, 7, 23, 15, 25

Section 5.1

3. $y''+4y=0,\ y_1=\cos 2x,\ y_2=\sin 2x;\ y(0)=3,\ y'(0)=8$
   - $y_1=\cos 2x \rightarrow y_1'=-2\sin 2x,\ y_1''=-4\cos 2x$
     Plug in: $-4\cos 2x+4\cos 2x=0$ ✓
   - $y_2=\sin 2x \rightarrow y_2'=2\cos 2x,\ y_2''=-4\sin 2x$
     Plug in: $-4\sin 2x+4\sin 2x=0$ ✓
   - General solution: $y=c_1\cos 2x+c_2\sin 2x$
   - Find c_1, c_2: $y'=-2c_1\sin 2x+2c_2\cos 2x$
     $y(0)=3 \Rightarrow 3=c_1$
     $y'(0)=8 \rightarrow 8=2c_2 \Rightarrow c_2=4$
     So $\boxed{y=3\cos 2x+4\sin 2x}$

16. $x^2y''+xy'+y=0,\ y_1=\cos(\ln x),\ y_2=\sin(\ln x);\ y(1)=2,\ y'(1)=3$
   - $y_1=\cos(\ln x) \rightarrow y_1'=-\frac{1}{x}\sin(\ln x),\ y_1''=\frac{1}{x^2}\sin(\ln x)-\frac{1}{x^2}\cos(\ln x)$
     Plug in: $x^2\left(\frac{1}{x^2}\sin(\ln x)-\frac{1}{x^2}\cos(\ln x)\right)+x\left(-\frac{1}{x}\sin(\ln x)\right)+\cos(\ln x)=0$ ✓
   - $y_2=\sin(\ln x) \rightarrow y_2'=\frac{1}{x}\cos(\ln x),\ y_2''=-\frac{1}{x^2}\cos(\ln x)-\frac{1}{x^2}\sin(\ln x)$
     Plug in: $x^2\left(-\frac{1}{x^2}\cos(\ln x)-\frac{1}{x^2}\sin(\ln x)\right)+x\left(\frac{1}{x}\cos(\ln x)\right)+\sin(\ln x)=0$ ✓
   - General solution: $y=c_1\cos(\ln x)+c_2\sin(\ln x)$
   - Find c_1, c_2: $y'=c_1\left(-\frac{1}{x}\sin(\ln x)\right)+c_2\left(\frac{1}{x}\cos(\ln x)\right)$
     $y(1)=2 \Rightarrow 2=c_1$
     $y'(1)=3 \Rightarrow 3=c_2$
     So $\boxed{y=2\cos(\ln x)+3\sin(\ln x)}$

20. $f(x)=\pi, g(x)=\cos^2 x+\sin^2 x$

$c_1\pi + c_2(\cos^2 x+\sin^2 x)=0$

$c_1\pi + c_2 = 0$

$c_1 = -c_2/\pi$

$\Rightarrow$ c_2 arbitrary, so infinitely many solutions

$\Rightarrow$ f, g are linearly dependent.

25. $f(x)=e^x\sin x$, $g(x)=e^x\cos x$

$c_1e^x\sin x + c_2e^x\cos x=0$

$e^x(c_1\sin x + c_2\cos x)=0$

$c_1\sin x + c_2\cos x = 0$

$c_1\sin x = -c_2\cos x$

True for all x only if $c_1=c_2=0$

$\Rightarrow$ f, g are linearly independent.

• Section 5.2

21. $y''+y=3x$; $y(0)=2$, $y'(0)=-2$

$y_c=c_1\cos x + c_2\sin x$; $y_p=3x$

• General solution: $y=y_c+y_p$

$\Rightarrow y=c_1\cos x + c_2\sin x + 3x$

• Find c_1, c_2: $y'=-c_1\sin x + c_2\cos x + 3$

$y(0)=2 \Rightarrow 2=c_1$

$y'(0)=-2 \Rightarrow -2=c_2+3 \Rightarrow c_2=-5$

Sol $\boxed{y=2\cos x - 5\sin x + 3x}$

7. $f(x)=1, g(x)=x, h(x)=x^2$

$$W(f,g,h)=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 1(1)(2)=2\neq 0$$

$\Rightarrow$ f, g, h are linearly independent.

16. $y^{(3)}-5y''+8y'-4y=0$, $y(0)=1, y'(0)=4, y''(0)=0$

$y_1=e^x, y_2=e^{2x}, y_3=xe^{2x}$

- General solution: $y=c_1e^x+c_2e^{2x}+c_3xe^{2x}$
- Find c_1,c_2,c_3: $y'=c_1e^x+2c_2e^{2x}+c_3e^{2x}+2c_3xe^{2x}$

$y''=c_1e^x+4c_2e^{2x}+4c_3e^{2x}+4c_3xe^{2x}$

$y(0)=1 \to 1=c_1+c_2$

$y'(0)=4 \Rightarrow 4=c_1+2c_2+c_3$

$y''(0)=0 \to 0=c_1+4c_2+4c_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{array}\right) \xrightarrow[r_3\to r_3-r_1]{r_2\to r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{array}\right) \xrightarrow{r_3\to r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{array}\right)$$

$c_3=-10$

$c_2+c_3=3 \to c_2=13$

$c_1+c_2=1 \to c_1=-12$

So $\boxed{y=-12e^x+13e^{2x}-10xe^{2x}}$

- Section 5.3

2. $2y''-3y'=0$

- characteristic equation: $2\lambda^2-3\lambda=0$

$\lambda(2\lambda-3)=0 \to \lambda_1=0, \lambda_2=3/2$

$y_1=1, y_2=e^{3/2x} \Rightarrow \boxed{y=c_1+c_2e^{3/2x}}$

4. $2y''-7y'+3y=0$

- characteristic equation: $2\lambda^2-7\lambda+3=0$

$(2\lambda-1)(\lambda-3)=0$

$\lambda_1=\frac{1}{2}, \lambda_2=3$

$y_1=e^{\frac{1}{2}x}, y_2=e^{3x} \to \boxed{y=c_1e^{\frac{1}{2}x}+c_2e^{3x}}$

7. $4y''-12y'+9y=0$

Characteristic equation: $4\lambda^2-12\lambda+9=0$

$(2\lambda-3)^2=0 \to \lambda_1=\lambda_2=3/2$

$$\boxed{y = c_1 e^{3/2 x} + c_2 x e^{3/2 x}}$$

23. $y'' - 6y' + 25y = 0,\ y(0) = 3,\ y'(0) = 1$

- General solution:
  - characteristic equation: $\lambda^2 - 6\lambda + 25 = 0$

$$\lambda = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm i\sqrt{64}}{2} = \frac{6 \pm 8i}{2}$$

$$\lambda = 3 \pm 4i$$

$\Rightarrow y = c_1 e^{3x}\cos(4x) + c_2 e^{3x}\sin(4x)$

- Find c_1, c_2: $y' = 3c_1 e^{3x}\cos(4x) - 4c_1 e^{3x}\sin(4x) + 3c_2 e^{3x}\sin(4x) + 4c_2 e^{3x}\cos(4x)$

$y(0) = 3 \Rightarrow 3 = c_1$

$y'(0) = 1 \Rightarrow 1 = 3c_1 + 4c_2 \Rightarrow 4c_2 = -8 \Rightarrow c_2 = -2$

So $\boxed{y = 3e^{3x}\cos(4x) - 2e^{3x}\sin(4x)}$

15. $y^{(4)} - 8y'' + 16y = 0$

- characteristic equation: $\lambda^4 - 8\lambda^2 + 16 = 0$

$$(\lambda^2 - 4)^2 = 0$$

$\lambda = \pm 2$ (double roots)

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} + c_3 x e^{-2x} + c_4 x e^{2x}}$$

25. $3y''' + 2y'' = 0,\ y(0) = -1,\ y'(0) = 0,\ y''(0) = 1$

- General solution:
  - characteristic equation: $3\lambda^3 + 2\lambda^2 = 0$

$\lambda^2(3\lambda + 2) = 0 \rightarrow \lambda_1 = \lambda_2 = 0,\ \lambda_3 = -2/3$

$y = c_1 + c_2 x + c_3 e^{-2/3 x}$

- Find c_1, c_2, c_3: $y' = c_2 - 2/3\, c_3 e^{-2/3 x}$

$y'' = (4/9) c_3 e^{-2/3 x}$

$y(0) = -1 \Rightarrow -1 = c_1 + c_3 \Rightarrow c_1 = -13/4$

$y'(0) = 0 \rightarrow 0 = c_2 - 2/3\, c_3 \Rightarrow c_2 = 3/2$

$y'(0) = 1 \rightarrow 1 = 4/9\, c_3 \rightarrow c_3 = 9/4$

So $\boxed{y = -\frac{13}{4} + \frac{3}{2}x + \frac{9}{4} e^{-2/3 x}}$