Suggested Problems Solutions - Sec 5.4, 5.6
Math 81
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- Section 5.4: 4, 22
- Section 5.6: 12

Section 5.4

4a) $mx'' + Kx = 0$

$m = 250g = 0.25\ kg$

$K \rightarrow F = -Kx$

$-9 = -K(0.25) \Rightarrow K = 36\ N/m$

$\Rightarrow 0.25x'' + 36x = 0,\ x(0) = 1,\ x'(0) = -5$

General solution

Characteristic equation: $0.25r^2 + 36 = 0$

$r^2 + 144 = 0$

$r = \pm 12i$

$\Rightarrow x = C_1\cos(12t) + C_2\sin(12t)$

Find C_1, C_2: $x' = -12C_1\sin(12t) + 12C_2\cos(12t)$

$x(0) = 1 \Rightarrow 1 = C_1$

$x'(0) = -5 \Rightarrow -5 = 12C_2 \Rightarrow C_2 = -5/12$

So $x(t) = \cos(12t) - \frac{5}{12}\sin(12t)$

Rewrite as $x(t) = C\cos(\omega_0 t - \alpha)$

$C = \sqrt{C_1^2 + C_2^2} = \sqrt{1^2 + (5/12)^2} = 13/12$

$\tan\alpha = C_2/C_1 = -5/12 \Rightarrow \alpha = \tan^{-1}(-5/12) + 2\pi$

$= 5.888$

So $x(t) = \frac{13}{12}\cos(12t - 5.888)$ m

b) Amplitude $= 13/12$ m

Period $= 2\pi/\omega_0 = \frac{2\pi}{12} = \pi/6.5$

22 $my'' + cy' + ky = 0$

$m = 0.375$

$c = 3$

Find k: $F = -ky \Rightarrow -12 = -k(0.5)$

$k = 24$

a) $\Rightarrow 0.375y'' + 3y' + 24y = 0,\ y(0) = 1,\ y'(0) = 0$

General solution

Characteristic equation: $0.375r^2 + 3r + 24 = 0$

$3r^2 + 24r + 192 = 0$

$r^2 + 8r + 64 = 0$

$r = \frac{-8 \pm \sqrt{64 - 4(64)}}{2} = \frac{-8 \pm 8\sqrt{3}\,i}{2}$

$r_{1,2} = -4 \pm 4\sqrt{3}\,i$

$y = c_1 e^{-4t}\cos(4\sqrt{3}t) + c_2 e^{-4t}\sin(4\sqrt{3}t)$

Find c_1, c_2: $y' = -4c_1 e^{-4t}\cos(4\sqrt{3}t) - 4\sqrt{3}c_2 e^{-4t}\sin(4\sqrt{3}t) - 4c_2 e^{-4t}\sin(4\sqrt{3}t) + 4\sqrt{3}c_2 e^{-4t}\cos(4\sqrt{3}t)$

$y(0) = 1 \Rightarrow 1 = c_1$

$y'(0) = 0 \Rightarrow 0 = -4c_1 + 4\sqrt{3}c_2 \Rightarrow c_2 = 4/4\sqrt{3} = \sqrt{3}/3$

$y = e^{-4t}\cos(4\sqrt{3}t) + \sqrt{3}/3\, e^{-4t}\sin(4\sqrt{3}t)$

b) Re-write $y = Ce^{-4t}\cos(4\sqrt{3}t - \alpha)$

$C = \sqrt{c_1^2 + c_2^2} = \sqrt{1^2 + (\sqrt{3}/3)^2} = 2\sqrt{3}/3$

$\alpha = \tan^{-1}(c_2/c_1) = \tan^{-1}(\sqrt{3}/3) = \pi/6$

$y = \frac{2\sqrt{3}}{3} e^{-4t}\cos(4\sqrt{3}t - \pi/6)$

- Amplitude: $(2\sqrt{3}/3)e^{-4t}$ ft.
- Frequency: $4\sqrt{3}$ s^{-1}
- Phase angle: $\pi/6$.

Section 5.6

12. $x''+6x'+13x=10\sin(5t),\ x(0)=x'(0)=0$

General solution

Homogeneous: $x_c''+6x_c'+13x_c=0$

characteristic equation: $r^2+6r+13=0$

$$r=\frac{-6\pm\sqrt{36-4(13)}}{2}=\frac{-6\pm\sqrt{-16}}{2}$$

$$r=-3\pm 2i$$

$$x_c=c_1e^{-3t}\cos(2t)+c_2e^{-3t}\sin(2t)$$

Particular: $x_p''+6x_p'+13x_p=10\sin(5t)$

Try $x_p=A\cos(5t)+B\sin(5t)$

$x_p'=-5A\sin(5t)+5B\cos(5t)$

$x_p''=-25A\cos(5t)-25B\sin(5t)$

Plug in: $-25A\cos(5t)-25B\sin(5t)+6(-5A\sin(5t)+5B\cos(5t))$
$+13(A\cos(5t)+B\sin(5t))=10\sin(5t)$

$\Rightarrow(-12A+30B)\cos(5t)+(-30A-12B)\sin(5t)=10\sin(5t)$

Equate coefficients: $-12A+30B=0\Rightarrow A=\frac{30}{12}B=\frac{5}{2}B$

$-30A-12B=10$

Solving: $A=\frac{-50}{174}$

$B=\frac{-10}{87}$

$$x_p=\frac{-50}{174}\cos(5t)-\frac{10}{87}\sin(5t)$$

$x=x_c+x_p\Rightarrow x=c_1e^{-3t}\cos(2t)+c_2e^{-3t}\sin(2t)-\frac{50}{174}\cos(5t)-\frac{10}{87}\sin(5t)$

Find c_1,c_2: $x'=-3c_1e^{-3t}\cos(2t)-2c_1e^{-3t}\sin(2t)-3c_2e^{-3t}\sin(2t)$
$+2c_2e^{-3t}\cos(2t)+\frac{250}{174}\sin(5t)-\frac{50}{87}\cos(5t)$

$x(0)=0\Rightarrow 0=c_1-\frac{50}{174}\Rightarrow c_1=\frac{50}{174}$

$x'(0)=0\Rightarrow 0=-3c_1+2c_2-\frac{50}{87}\Rightarrow c_2=\frac{125}{174}$

So $x=\frac{50}{174}e^{-3t}\cos(2t)+\frac{125}{174}e^{-3t}\sin(2t)-\frac{50}{174}\cos(5t)-\frac{10}{87}\sin(5t)$

$$x_{tr}=\frac{50}{174}e^{-3t}\cos(2t)+\frac{125}{174}e^{-3t}\sin(2t)$$

$C=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{50}{174}\right)^2+\left(\frac{125}{174}\right)^2}=\frac{25}{6\sqrt{29}}$

$\alpha=\tan^{-1}(c_2/c_1)=\tan^{-1}\left(\frac{125/174}{50/174}\right)=\tan^{-1}\left(\frac{5}{2}\right)\approx 1.1903$

$$\Rightarrow\boxed{x_{tr}(t)=\frac{25}{6\sqrt{29}}e^{-3t}\cos(3t-1.1903)}$$

$x_{sp}(t) = -\frac{50}{174}\cos(5t) - \frac{10}{87}\sin(5t)$

$C = \sqrt{(-\frac{50}{174})^2 + (-\frac{10}{87})^2} = 5/(3\sqrt{29})$

$\alpha = \pi + \tan^{-1}(-\frac{10}{87} / -\frac{50}{174}) = \pi + \tan^{-1}(\frac{2}{5}) \approx 3.5221$

So $x_{sp}(t) = \frac{5}{3\sqrt{29}}\cos(5t - 3.5221)$