Solutions to Suggested Problems - Section 5.5
Math 81, Dr. Doreen DeLeon

- Section 5.5: 32, 33, 39, 53, 54

32. $y''+3y'+2y=e^x,\ y(0)=0,\ y'(0)=3$

- General solution
  - Homogeneous: $y_c''+3y_c'+2y_c=0$

    Characteristic equation: $r^2+3r+2=0$

    $(r+1)(r+2)=0 \Rightarrow r_1=-1, r_2=-2$

    $y_c=c_1e^{-x}+c_2e^{-2x}$
  - Particular: $y_p''+3y_p'+2y_p=e^x$

    Try $y_p=Ae^x$

    $y_p'=Ae^x$

    $y_p''=Ae^x$

    Plug in: $Ae^x+3Ae^x+2Ae^x=e^x$

    $6Ae^x=e^x$

    $6A=1 \Rightarrow A=1/6$

    So $y_p=\frac{1}{6}e^x$

  $y=y_c+y_p \Rightarrow y=c_1e^{-x}+c_2e^{-2x}+\frac{1}{6}e^x$
- Find c_1,c_2: $y'=-c_1e^{-x}-2c_2e^{-2x}+\frac{1}{6}e^x$

  $y(0)=0 \Rightarrow 0=c_1+c_2+1/6 \rightarrow c_1+c_2=-1/6$

  $y'(0)=3 \Rightarrow 3=-c_1-2c_2+1/6 \quad -c_1-2c_2=17/6$

  $-c_2=16/6=8/3 \Rightarrow c_2=-8/3$

  $c_1=-\frac{1}{6}-c_2=-\frac{1}{6}+8/3=15/6=5/2$

$$\boxed{y=\left(\tfrac{5}{2}\right)e^{-x}-\left(8/3\right)e^{-2x}+\left(1/6\right)e^x}$$

33. $y''+9y=\sin(2x),\ y(0)=1,\ y'(0)=0$

- General solution
  - Homogeneous: $y_c''+9y_c=0$

    Characteristic equation: $r^2+9=0$

    $r_{1,2}=\pm 3i$

    $y_c=c_1\cos(3x)+c_2\sin(3x)$

(over ⇒)

Mead

• Particular: $y_p'' + 9y_p = \sin 2x$

Try $y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

Plug in: $-4A\cos 2x - 4B\sin 2x + 9(A\cos 2x + B\sin 2x) = \sin 2x$

$5A\cos 2x + 5B\sin 2x = \sin 2x$

Equate coefficients: $5A = 0 \rightarrow A = 0$

$5B = 1 \Rightarrow B = \frac{1}{5}$

So $y_p = \frac{1}{5}\sin 2x$

$y = y_c + y_p \Rightarrow y = c_1\cos 3x + c_2\sin 3x + \frac{1}{5}\sin 2x$

• Find c_1 and c_2: $y' = -3c_1\sin 3x + 3c_2\cos 3x + \frac{2}{5}\cos 2x$

$y(0) = 1 \rightarrow 1 = c_1$

$y'(0) = 0 \Rightarrow 0 = 3c_2 + \frac{2}{5} \Rightarrow c_2 = \frac{2}{15}$

So $\boxed{y = \cos 3x - (\frac{2}{15})\sin 3x + (\frac{1}{5})\sin 2x}$

39. $y''' + y'' = x + e^{-x}$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

• General solution

• Homogeneous: $y_c''' + y_c'' = 0$

Characteristic equation: $r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r = 0$ (double root), -1

$y_c = c_1 + c_2 x + c_3 e^{-x}$

• Particular: split into 2 parts

$y_{p_1}''' + y_{p_1}'' = x$

Try $y_{p_1} = \cancel{(Ax+B)}(Ax+B)x = \cancel{Ax^2+Bx}$

$= (Ax+B)x^2 = Ax^3 + Bx^2$

$y_{p_1}' = 3Ax^2 + 2Bx$

$y_{p_1}'' = 6Ax + 2B$

$y_{p_1}''' = 6A$

Plug in: $6A + 1(6Ax + 2B) = x$

$6Ax + 6A + 2B = x$

Equate coeffs: $6A = 1 \Rightarrow A = \frac{1}{6}$

$6A + 2B = 0 \Rightarrow B = -\frac{1}{2}$

$y_{p_2}''' + y_{p_2}'' = e^{-x}$

Try $y_{p_2} = \cancel{Ae^{-x}}\ Axe^{-x}$

$y_{p_2}' = Ae^{-x} - Axe^{-x}$

$y_{p_2}'' = -2Ae^{-x} + Axe^{-x}$

$y_{p_2}''' = 3Ae^{-x} - Axe^{-x}$

Plug in:

$3Ae^{-x} - Axe^{-x} + (-2Ae^{-x} + Axe^{-x}) = e^{-x}$

$Ae^{-x} = e^{-x} \Rightarrow A = 1$

So $y_{p_1} = \frac{1}{6}x^3 - \frac{1}{2}x^2$, $y_{p_2} = xe^{-x}$

$y_p = y_{p_1} + y_{p_2} \Rightarrow y_p = \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$

$y = y_c + y_p \Rightarrow y = c_1 + c_2x + c_3e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$

• Find c_1, c_2, c_3: $y' = c_2 - c_3e^{-x} + \frac{1}{2}x^2 - x + e^{-x} - xe^{-x}$

$y'' = c_3e^{-x} + x - 1 - 2e^{-x} + xe^{-x}$

$y(0) = 1 \Rightarrow 1 = c_1 + c_3 \Rightarrow c_1 = 1 - c_3 = -3$

$y'(0) = 0 \Rightarrow 0 = c_2 - c_3 + 1 \Rightarrow c_2 - c_3 = -1 \Rightarrow c_2 = -1 + c_3 = 3$

$y''(0) = 1 \Rightarrow 1 = c_3 - 1 - 2 \Rightarrow c_3 = 4$

So $\boxed{y = -3 + 3x + 4e^{-x} + (\tfrac{1}{6})x^3 - (\tfrac{1}{2})x^2 + xe^{-x}}$

53. $y'' + 9y = 2\sec 3x$

• Homogeneous: $y_c'' + 9y_c = 0$

$\Rightarrow y_c = c_1\cos 3x + c_2\sin 3x$ (see # 33)

• Particular: $y_p'' + 9y_p = 2\sec 3x$

Must use variation of parameters.

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}\,dx$$

$y_1 = \cos 3x$, $y_2 = \sin 3x$, $f(x) = 2\sec 3x$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$= 3\cos^2 3x + 3\sin^2 3x$

$= 3(\cos^2 3x + \sin^2 3x) = 3$

$$\text{So } y_p = -\cos 3x\int \frac{\sin 3x\,(2\sec 3x)}{3}\,dx + \sin 3x\int \frac{\cos 3x\,(2\sec 3x)}{3}\,dx$$

$$= -\frac{2}{3}\cos 3x\int \frac{\sin 3x}{\cos 3x}\,dx + \frac{2}{3}\sin 3x\int 1\,dx$$

$$= \frac{2}{9}\cos 3x\ln|\cos 3x| + \frac{2}{3}x\sin 3x$$

$$y = y_c + y_p \Rightarrow \boxed{y = c_1\cos 3x + c_2\sin 3x + \frac{2}{9}\cos 3x\ln|\cos 3x| + \frac{2}{3}x\sin 3x}$$

54. $y'' + y = \csc^2 x$

- Homogeneous: $y_c'' + y_c = 0$
  - characteristic equation: $r^2 + 1 = 0$

    $r = \pm i$

    $y_c = c_1 \cos x + c_2 \sin x$

- Particular: $y_p'' + y_p = \csc^2 x$

  Must use variation of parameters

  $$y_p = -y_1(x)\int \frac{y_2(x) f(x)}{W(y_1, y_2)}\,dx + y_2(x)\int \frac{y_1(x) f(x)}{W(y_1, y_2)}\,dx$$

  $y_1(x) = \cos x,\ y_2(x) = \sin x,\ f(x) = \csc^2 x$

  $$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin x(\csc^2 x)}{1}\,dx + \sin x \int \frac{\cos x(\csc^2 x)}{1}\,dx$$

$$= -\cos x \int \csc x\,dx + \sin x \int \cos x \sin^{-2} x\,dx$$

$$= -\cos x \cdot \ln|\csc x - \cot x| + \sin x(-\sin^{-1} x)$$

$$= -\cos x \ln|\csc x - \cot x| - 1$$

$$y = y_c + y_p \Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x - \cos x \ln|\csc x - \cot x| - 1}$$

Mead