Solution to Suggested Problems - Sec 6.1, 7.1-7.2
Math 81
Instructor: Dr. Doreen De Leon

- Section 6.1: 3, 13, 22
- Section 7.1: 1, 3
- Section 7.2: 3, 6, 16, 25

- <u>Section 6.1</u>

3. $A = \begin{pmatrix} 8 & -6 \\ 3 & -1 \end{pmatrix}$

- Eigenvalues: $0 = \det(A - \lambda I)$

$$= \det\left(\begin{pmatrix} 8 & -6 \\ 3 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 8-\lambda & -6 \\ 3 & -1-\lambda \end{pmatrix}$$

$$= \begin{vmatrix} 8-\lambda & -6 \\ 3 & -1-\lambda \end{vmatrix} = (8-\lambda)(-1-\lambda) + 18$$
$$= \lambda^2 - 7\lambda + 10$$
$$= (\lambda-2)(\lambda-5)$$

$$\lambda_1 = 2, \lambda_2 = 5$$

- Eigenvectors:

$\underline{\lambda_1 = 2}$ Solve $(A - \lambda_1 I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 8-2 & -6 \\ 3 & -1-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \xrightarrow{\frac{1}{6}r_1} \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

v_2 arbitrary, so let $v_2 = t$

$v_1 - v_2 = 0 \to v_1 = t$

$$\vec{v} = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $\boxed{\lambda_1 = 2, \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\boxed{\lambda_2 = 5}$ Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \xrightarrow{r_1 \to \frac{1}{3}r_1} \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & - \\ 0 & \end{pmatrix}$$

v_2 arbitrary, so let $v_2 = t$

$v_1 - 2v_2 = 0 \rightarrow v_1 = 2t \quad \Rightarrow \vec{v} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So $\boxed{\lambda_2 = 5,\ \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

13. $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{pmatrix}$

- Eigenvalues: $0 = \det(A - \lambda I)$

$$= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 2 & -2-\lambda & -1 \\ -2 & 6 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 6 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)[(-2-\lambda)(3-\lambda)+6]$$

$$= (2-\lambda)[\lambda^2 - \lambda]$$

$$= \lambda(\lambda-1)(2-\lambda) \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2$$

- Eigenvectors:

$\boxed{\lambda_1 = 0}$ Solve $A\vec{v} = \vec{0}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{pmatrix} \xrightarrow[r_3 \to r_3 + r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 3r_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow v_3$ arbitrary + $2v_1 = 0 \Rightarrow v_1 = 0$

$-2v_2 - v_3 = 0 \Rightarrow v_2 = -\frac{1}{2}v_3$

let $v_3 = 2t \Rightarrow v_2 = -t$

So $\vec{v} = \begin{pmatrix} 0 \\ -t \\ 2t \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \Rightarrow \boxed{\lambda_1 = 0,\ \vec{v} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}}$

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$\boxed{\lambda_2 = 1}$ Solve $(A-I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -1 \\ -2 & 6 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -1 \\ -2 & 6 & 2 \end{pmatrix} \xrightarrow[r_3 \to r_3 + 2r_1]{r_2 \to r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 6 & 2 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 2r_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad v_1 = 0$

$-3v_2 - v_3 = 0 \Rightarrow v_2 = -\frac{1}{3}v_3$

v_3 arbitrary, so let $v_3 = 3t$

$\Rightarrow v_2 = -t$

So $\vec{v} = \begin{pmatrix} 0 \\ -t \\ 3t \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \boxed{\lambda_2 = 1, \vec{v} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}}$

$\boxed{\lambda_3 = 2}$ Solve $(A-2I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -1 \\ -2 & 6 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -1 \\ -2 & 6 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} -2 & 6 & 1 \\ 2 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \to r_2 + r_1} \begin{pmatrix} -2 & 6 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$-2v_1 + 6v_2 + v_3 = 0$

$2v_2 = 0 \rightarrow v_2 = 0$

$\Rightarrow v_1 = \frac{1}{2}v_3$

v_3 arbitrary, so ~~let $v_3 = 2t \Rightarrow v_1 = t$~~

So $\vec{v} = \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \Rightarrow \boxed{\lambda_3 = 2, \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$

$\boxed{\text{over} \Rightarrow}$

22. $A = \begin{pmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{pmatrix}$

- Eigenvalues: $0 = \det(A - \lambda I)$

$$= \begin{vmatrix} 5-\lambda & -6 & 3 \\ 6 & -7-\lambda & 3 \\ 6 & -6 & 2-\lambda \end{vmatrix} \begin{matrix} 5-\lambda & -6 \\ 6 & -7-\lambda \\ 6 & -6 \end{matrix}$$

$$= (5-\lambda)(-7-\lambda)(2-\lambda) - 108 - 108$$
$$-6(-7-\lambda)(3) + 18(5-\lambda) + 36(2-\lambda)$$
$$= (5-\lambda)(-7-\lambda)(2-\lambda) - 216 - 18(-7-\lambda) + 18(5-\lambda)$$
$$+36(2-\lambda)$$
$$= (5-\lambda)(\lambda^2 + 5\lambda - 14) - 216 + 126 + \cancel{18\lambda} + 90 - \cancel{18\lambda}$$
$$+ 72 - 36\lambda$$
$$= -\lambda^3 + 39\lambda - 70 + 72 - 36\lambda$$
$$= -\lambda^3 + 3\lambda + 2$$
$$= (\lambda+1)(-\lambda^2 + \lambda + 2)$$
$$= (\lambda+1)(\lambda+1)(\lambda-2) \Rightarrow \lambda_1 = -1 \text{ (multiplicity 2)}, \lambda_2 = 2$$

- Eigenvectors

$\boxed{\lambda_1 = -1}$ Solve $(A+I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 6 & -6 & 3 \\ 6 & -6 & 3 \\ 6 & -6 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -6 & 3 \\ 6 & -6 & 3 \\ 6 & -6 & 3 \end{pmatrix} \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 6 & -6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} 6v_1 - 6v_2 + 3v_3 = 0 \\ v_1 = v_2 - \frac{1}{2}v_3 \end{matrix}$$

v_2, v_3 arbitrary, so let $v_2 = s$, $v_3 = 2t$
$\Rightarrow v_1 = s - t$

$$\Rightarrow \vec{v} = \begin{pmatrix} s-t \\ s \\ 2t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\boxed{\lambda_1 = -1, \text{ basis for eigenspace} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}}$$

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$\boxed{\lambda_2=2}$ Solve $(A-2I)\vec{v}=\vec{0}$

$$\begin{pmatrix}3&-6&3\\6&-9&3\\6&-6&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}3&-6&3\\6&-9&3\\6&-6&0\end{pmatrix}\xrightarrow[r_3\to r_3-2r_1]{r_2\to r_2-2r_1}\begin{pmatrix}3&-6&3\\0&3&-3\\0&6&-6\end{pmatrix}\xrightarrow{r_3\to r_3-2r_2}\begin{pmatrix}3&-6&3\\0&3&-3\\0&0&0\end{pmatrix}$$

$3v_1-6v_2+3v_3=0 \Rightarrow v_1=2v_2-v_3$

$3v_2-3v_3=0 \Rightarrow v_2=v_3$

v_3 arbitrary, so let $v_3=t \Rightarrow v_2=t,\ v_1=t$

So, $\vec{v}=\begin{pmatrix}t\\t\\t\end{pmatrix}=t\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \boxed{\lambda_2=2,\ \vec{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$

- Section 7.1

1. $x''+3x'+7x=t^2$

Let $x_1=x$

$x_2=x'$

Then $x_1'=x'$. But $x'=x_2 \Rightarrow x_1'=x_2$

$x_2'=x''$. But, since $x''+3x'+7x=t^2$, $x''=-3x'-7x+t^2$

$=-3x_2-7x_1+t^2$

$\Rightarrow x_2'=-3x_2-7x_1+t^2$

So, $\boxed{\begin{aligned}x_1'&=x_2\\x_2'&=-7x_1-3x_2+t^2\end{aligned}}$

3. $t^2x''+tx'+(t^2-1)x=0$

Let $x_1=x$

$x_2=x'$

Then $x_1'=x'=x_2$

$x_2'=x''$,

$t^2x''+tx'+(t^2-1)x=0 \Rightarrow x''=-\frac{1}{t}x'-\frac{t^2-1}{t^2}x$

$=-\frac{1}{t}x_2-\left(\frac{t^2-1}{t^2}\right)x_1$

So $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{(t^2-1)}{t^2}x_1 - \frac{1}{t}x_2 \end{aligned}}$

Section 7.2

3. $x' = -3y,\ y' = -3x$

$\Rightarrow \begin{aligned} x' &= 0x - 3y \\ y' &= -3x + 0y \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boxed{\vec{x}' = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\vec{x}}$

6. $x' = tx - e^t y + \cos t,\ y' = e^{-t}x + t^2 y - \sin t$

$\begin{aligned} x' &= tx - e^t y + \cos t \\ y' &= e^{-t}x + t^2 y - \sin t \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} t & -e^t \\ e^{-t} & t^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\boxed{\vec{x}' = \begin{pmatrix} t & -e^t \\ e^{-t} & t^2 \end{pmatrix}\vec{x} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}$

16. $\vec{x}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\vec{x},\ \vec{x}_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \vec{x}_2 = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix},\ \vec{x}_2 = \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix}$

- $\vec{x}_1' = \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \end{pmatrix};\ \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\vec{x}_1 = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} = \begin{pmatrix} 4e^{3t} - e^{3t} \\ -2e^{3t} - e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \end{pmatrix} = \vec{x}_1'$ ✓
- $\vec{x}_2' = \begin{pmatrix} 2e^{2t} \\ -4e^{2t} \end{pmatrix};\ \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\vec{x}_2 = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix} = \begin{pmatrix} 4e^{2t} - 2e^{2t} \\ -2e^{2t} - 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -4e^{2t} \end{pmatrix} = \vec{x}_2'$ ✓

$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = e^{3t}(-2e^{2t}) - (-e^{3t})e^{2t}$

$= -2e^t + e^t = -e^t \neq 0 \Rightarrow \{\vec{x}_1, \vec{x}_2\}$ lin. indep.

$\Rightarrow \vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 \Rightarrow \vec{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

25. Find c_1, c_2 in #16 given $x_1(0) = 11,\ x_2(0) = -7$

$\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix} \Rightarrow \begin{pmatrix} 11 \\ -7 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow \begin{aligned} c_1 + c_2 &= 11 \\ -c_1 - 2c_2 &= -7 \end{aligned} \Big\rangle \begin{aligned} c_1 &= 15 \\ c_2 &= -4 \end{aligned} \Rightarrow \boxed{\vec{x}(t) = 15e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 4e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$