Solutions to Suggested Problems - Sec 7.3 + 7.5
Math 81
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- Section 7.3: 3, 21, 9, 13 (omit graphs)
- Section 7.5: 5, 9

- Section 7.3

3 $x_1' = 3x_1 + 4x_2$, $x_2' = 3x_1 + 2x_2$; $x_1(0) = x_2(0) = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- General solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 5\lambda - 6$$
$$= (\lambda - 6)(\lambda + 1)$$
$$\Rightarrow \lambda_1 = -1, \lambda_2 = 6$$

$\boxed{\lambda_1 = -1}$ Solve $(A - (-1)I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{r_2 \to r_2 - \frac{3}{4} r_1} \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \Rightarrow 4u_1 + 4u_2 = 0 \Rightarrow u_1 = -u_2$$

u_2 arbitrary, so let $u_2 = 1$
$\rightarrow u_1 = -1$

So $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = e^{\lambda_1 t}\vec{u} = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{\lambda_2 = 6}$ Solve $(A - 6I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \xrightarrow{r_2 \to r_2 + r_1} \begin{pmatrix} -3 & 4 \\ 0 & 0 \end{pmatrix} \Rightarrow -3v_1 + 4v_2 = 0 \Rightarrow v_1 = \frac{4}{3} v_2$$

v_2 arbitrary, so let $v_2 = 3 \Rightarrow v_1 =$

So $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \Rightarrow \vec{x}_2 = e^{\lambda_2 t}\vec{v} = e^{6t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

general solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

$$\Rightarrow \vec{x} = c_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{6t}\begin{pmatrix}4\\3\end{pmatrix}$$

Find c_1 and c_2: $\vec{x}(0) = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}-1\\1\end{pmatrix} + c_2\begin{pmatrix}4\\3\end{pmatrix}$

$$\Rightarrow \begin{aligned} -c_1 + 4c_2 &= 1 \\ c_1 + 3c_2 &= 1 \end{aligned}$$

$$7c_2 = 2 \Rightarrow c_2 = 2/7$$

$$c_1 = 1 - 3c_2 \Rightarrow c_1 = 1/7$$

So $\vec{x} = \frac{1}{7}e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + \frac{2}{7}e^{6t}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-\frac{1}{7}e^{-t} + 8/7\, e^{6t}\\ \frac{1}{7}e^{-t} + 6/7\, e^{6t}\end{pmatrix}$

$$\Rightarrow \boxed{x_1 = -\tfrac{1}{7}e^{-t} + 8/7\, e^{6t},\ x_2 = \tfrac{1}{7}e^{-t} + 6/7\, e^{6t}}$$

21. $x_1' = 5x_1 - 6x_3,\ x_2' = 2x_1 - x_2 - 2x_3,\ x_3' = 4x_1 - 2x_2 - 4x_3$

$$\vec{x}' = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}' = \begin{pmatrix}5 & 0 & -6\\2 & -1 & -2\\4 & -2 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \underbrace{\begin{pmatrix}5 & 0 & -6\\2 & -1 & -2\\4 & -2 & -4\end{pmatrix}}_{= A}\vec{x}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix}5-\lambda & 0 & -6\\2 & -1-\lambda & -2\\4 & -2 & -4-\lambda\end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix}-1-\lambda & -2\\-2 & -4-\lambda\end{vmatrix} - 6\begin{vmatrix}2 & -1-\lambda\\4 & -2\end{vmatrix}$$

$$= (5-\lambda)((-1-\lambda)(-4-\lambda) - 4) - 6(-4 - 4(-1-\lambda))$$

$$= (5-\lambda)(\lambda^2 + 5\lambda) - 6(4\lambda)$$

$$= -\lambda^3 + 25\lambda - 24\lambda$$

$$= -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = \lambda(\lambda+1)(\lambda-1)$$

$$\Rightarrow \lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 1$$

$\lambda_1 = -1$ / Solve $(A - (-1)I)\vec{u} = \vec{0}$

$$\begin{pmatrix}6 & 0 & -6\\2 & 0 & -2\\-4 & 2 & -3\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

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$$\begin{pmatrix}6&0&-6\\2&0&-2\\-4&-2&-3\end{pmatrix}\xrightarrow{r_1\to\frac{1}{6}r_1}\begin{pmatrix}1&0&-1\\2&0&-2\\-4&-2&-3\end{pmatrix}\xrightarrow[r_3\to r_3+4r_1]{r_2\to r_2-2r_1}\begin{pmatrix}1&0&-1\\0&0&0\\0&-2&1\end{pmatrix}\xrightarrow{r_2\leftrightarrow r_3}$$

$$\begin{pmatrix}1&0&-1\\0&-2&1\\0&0&0\end{pmatrix}\Rightarrow \begin{array}{l} u_1-u_3=0\to u_1=u_3\\ -2u_2+u_3=0\to u_2=\frac{1}{2}u_3\end{array}$$

u_3 arbitrary, so let $u_3=2\Rightarrow u_2=1, u_1=2$

So $\vec{u}=\begin{pmatrix}2\\1\\2\end{pmatrix}\Rightarrow \vec{x}_1=e^{\lambda_1 t}\vec{u}=e^{-t}\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\boxed{\lambda_2=0}$ Solve $A\vec{v}=\vec{0}$

$$\begin{pmatrix}5&0&-6\\2&-1&-2\\4&-2&-4\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}5&0&-6\\2&-1&-2\\4&-2&-4\end{pmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{pmatrix}2&-1&-2\\5&0&-6\\4&-2&-4\end{pmatrix}\xrightarrow[r_3\to r_3-2r_1]{r_2\to r_2-\frac{5}{2}r_1}\begin{pmatrix}2&-1&-2\\0&5/2&-1\\0&0&0\end{pmatrix}$$

$\Rightarrow 2v_1-v_2-2v_3=0 \to v_1=\frac{1}{2}(v_2)+v_3$

$5/2\, v_2-v_3=0 \to v_2=2/5\, v_3$

v_3 arbitrary, so let $v_3=5\Rightarrow v_2=2,\ v_1=\frac{1}{2}(2)+5=6$

So $\vec{v}=\begin{pmatrix}6\\2\\5\end{pmatrix}\Rightarrow \vec{x}_2=e^{\lambda_2 t}\vec{v}=\begin{pmatrix}6\\2\\5\end{pmatrix}$

$\boxed{\lambda_3=1}$ Solve $(A-I)\vec{w}=\vec{0}$

$$\begin{pmatrix}4&0&-6\\2&-2&-2\\4&-2&-5\end{pmatrix}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}4&0&-6\\2&-2&-2\\4&-2&-5\end{pmatrix}\xrightarrow[r_3\to r_3-r_1]{r_2\to r_2-\frac{1}{2}r_1}\begin{pmatrix}4&0&-6\\0&-2&1\\0&-2&1\end{pmatrix}\xrightarrow{r_3\to r_3-r_2}\begin{pmatrix}4&0&-6\\0&-2&1\\0&0&0\end{pmatrix}$$

$\Rightarrow 4w_1-6w_3=0\Rightarrow w_1=3/2\, w_3$

$-2w_2+w_3=0\Rightarrow w_2=\frac{1}{2}w_3$

w_3 arbitrary, so let $w_3=2\to w_2=1, w_1=3$

So $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \vec{x}_3 = e^{\lambda_3 t}\vec{w} = e^{t}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

General solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

$$\Rightarrow \boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} + c_3 e^{t}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}}$$

9. $x_1' = 2x_1 - 5x_2,\ x_2' = 4x_1 - 2x_2;\ x_1(0) = 2,\ x_2(0) = 3$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \vec{x}' = \underbrace{\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}}_{A}\vec{x},\ \vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

• General solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 20$$
$$= \lambda^2 - 4 + 20$$
$$= \lambda^2 + 16$$
$$\Rightarrow \lambda = \pm 4i$$

$\lambda = 4i$ | Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $(2-4i)v_1 - 5v_2 = 0 \Rightarrow v_1 = \frac{5}{2-4i}v_2$

v_2 arbitrary, so let $v_2 = 2-4i \Rightarrow v_1 = 5 \Rightarrow \vec{v} = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$

$$e^{\lambda t}\vec{v} = e^{4it}\begin{pmatrix} 5 \\ 2-4i \end{pmatrix} = (\cos(4t) + i\sin(4t))\begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$$
$$= \begin{pmatrix} 5(\cos(4t) + i\sin(4t)) \\ (2-4i)(\cos(4t) + i\sin(4t)) \end{pmatrix}$$
$$= \begin{pmatrix} 5\cos(4t) + i5\sin(4t) \\ 2\cos(4t) - 4i\cos(4t) + 2i\sin(4t) + 4\sin(4t) \end{pmatrix}$$
$$= \begin{pmatrix} 5\cos(4t) + i5\sin(4t) \\ 2\cos(4t) + 4\sin(4t) + i(-4\cos(4t) + 2\sin(4t)) \end{pmatrix}$$

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$$\Rightarrow e^{\lambda t}\vec{v} = \underbrace{\begin{pmatrix} 5\cos(4t) \\ 2\cos(4t)+4\sin(4t) \end{pmatrix}}_{= \vec{x}_1} + i\underbrace{\begin{pmatrix} 5\sin(4t) \\ -4\cos(4t)+2\sin(4t) \end{pmatrix}}_{= \vec{x}_2}$$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\Rightarrow \vec{x} = c_1\begin{pmatrix} 5\cos(4t) \\ 2\cos(4t)+4\sin(4t) \end{pmatrix} + c_2\begin{pmatrix} 5\sin(4t) \\ -4\cos(4t)+2\sin(4t) \end{pmatrix}$$

- Find c_1 and c_2:

$$\vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

$$\Rightarrow 5c_1 = 2 \Rightarrow c_1 = 2/5$$

$$2c_1 - 4c_2 = 3 \Rightarrow c_2 = \tfrac{1}{4}(2c_1 - 3) = \tfrac{-11}{20}$$

So $$\boxed{\vec{x}(t) = \frac{2}{5}\begin{pmatrix} 5\cos(4t) \\ 2\cos(4t)+4\sin(4t) \end{pmatrix} - \frac{11}{20}\begin{pmatrix} 5\sin(4t) \\ -4\cos(4t)+2\sin(4t) \end{pmatrix}}$$

13. $x_1' = 5x_1 - 9x_2,\ x_2' = 2x_1 - x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}' = \underbrace{\begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}}_{= A}\vec{x}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) + 18$$

$$= \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$\lambda = 2+3i$ | Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-(2+3i) & -9 \\ 2 & -1-(2+3i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3-3i & -9 \\ 2 & -3-3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $(3-3i)v_1 - 9v_2 = 0 \Rightarrow v_1 = \frac{9}{3-3i}v_2 = \frac{3}{1-i}v_2$

v_2 arbitrary, so let $v_2 = 1-i \Rightarrow v_1 = 3 \rightarrow \vec{v} = \begin{pmatrix} 3 \\ 1-i \end{pmatrix}$

$$e^{\lambda t}\vec{v} = e^{(2+3i)t}\begin{pmatrix}3\\1-i\end{pmatrix} = e^{2t}(\cos(3t)+i\sin(3t))\begin{pmatrix}3\\1-i\end{pmatrix}$$

$$= e^{2t}\begin{pmatrix}3(\cos(3t)+i\sin(3t))\\(1-i)(\cos(3t)+i\sin(3t))\end{pmatrix}$$

$$= e^{2t}\begin{pmatrix}3\cos(3t)+i(3\sin(3t))\\\cos(3t)+\sin(3t)+i(-\cos(3t)+\sin(3t))\end{pmatrix}$$

$$= \underbrace{e^{2t}\begin{pmatrix}3\cos(3t)\\\cos(3t)+\sin(3t)\end{pmatrix}}_{\vec{x}_1} + i\underbrace{e^{2t}\begin{pmatrix}3\sin(3t)\\-\cos(3t)+\sin(3t)\end{pmatrix}}_{\vec{x}_2}$$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\Rightarrow \boxed{\vec{x} = c_1 e^{2t}\begin{pmatrix}3\cos(3t)\\\cos(3t)+\sin(3t)\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}3\sin(3t)\\-\cos(3t)+\sin(3t)\end{pmatrix}}$$

Section 7.5

5. $\vec{x}\,' = \underbrace{\begin{pmatrix}7 & 1\\-4 & 3\end{pmatrix}}_{A}\vec{x}$

$$0 = \det(A-\lambda I) = \begin{vmatrix}7-\lambda & 1\\-4 & 3-\lambda\end{vmatrix} = (7-\lambda)(3-\lambda)+4$$
$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda-5)^2$$

$\Rightarrow \lambda = 5$ (algebraic multiplicity 2)

$\boxed{\lambda=5}$ Solve $(A-5I)\vec{v} = \vec{0}$

$$\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix} \xrightarrow{r_2 \to r_2 + 2r_1} \begin{pmatrix}2 & 1\\0 & 0\end{pmatrix} \Rightarrow 2v_1 + v_2 = 0 \to v_1 = -\tfrac{1}{2}v_2$$

v_2 arbitrary so let $v_2 = 2 \Rightarrow v_1 = -1$

$\Rightarrow \vec{v} = \begin{pmatrix}-1\\2\end{pmatrix}$. So one solution is $\vec{x}_1 = e^{\lambda t}\vec{v} = e^{5t}\begin{pmatrix}-1\\2\end{pmatrix}$.

Find a second solution, $\vec{x}_2 = e^{5t}(t\vec{v}+\vec{w}) = e^{5t}\left(t\begin{pmatrix}-1\\2\end{pmatrix}+\vec{w}\right)$

where $\vec{w}$ solves $(A-5I)\vec{w} = \vec{v}$

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$$\Rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ -4 & 2 & 2 \end{array}\right) \xrightarrow{r_2 \to r_2 + 2r_1} \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 2w_1 + w_2 = -1$$

w_2 arbitrary, so let $w_2 = 0 \Rightarrow 2w_1 = -1$
$w_1 = -\frac{1}{2}$

$$\Rightarrow \vec{w} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

So $\vec{x}_2 = e^{5t}\left(t\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}\right) = e^{5t}\begin{pmatrix} -t - \frac{1}{2} \\ 2t \end{pmatrix}$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 \Rightarrow \boxed{\vec{x} = c_1 e^{5t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} -t - \frac{1}{2} \\ 2t \end{pmatrix}}$$

9. $\vec{x}' = \underbrace{\begin{pmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{pmatrix}}_{A}\vec{x}$

$$0 = \det(A - \lambda I) = \begin{vmatrix} -19-\lambda & 12 & 84 \\ 0 & 5-\lambda & 0 \\ -8 & 4 & 33-\lambda \end{vmatrix} = (5-\lambda)(-1)^{2+2}\begin{vmatrix} -19-\lambda & 84 \\ -8 & 33-\lambda \end{vmatrix}$$

$$= (5-\lambda)((-19-\lambda)(33-\lambda) + 8(84))$$
$$= (5-\lambda)(\lambda^2 - 14\lambda + 45)$$
$$= (5-\lambda)(\lambda-5)(\lambda-9)$$

$\Rightarrow \lambda = 5$ (algebraic multiplicity 2), 9

$\lambda = 5$ | Solve $(A - 5I)\vec{u} = \vec{0}$

$$\begin{pmatrix} -24 & 12 & 84 \\ 0 & 0 & 0 \\ -8 & 4 & 28 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -24 & 12 & 84 \\ 0 & 0 & 0 \\ -8 & 4 & 28 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} -24 & 12 & 84 \\ -8 & 4 & 28 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 \to \frac{1}{3}r_1} \begin{pmatrix} -8 & 4 & 28 \\ -8 & 4 & 28 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \to r_2 - r_1}$$

$\boxed{\text{over}}$

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$$\begin{pmatrix} -8 & 4 & 28 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} -8u_1 + 4u_2 + 28u_3 = 0 = u_1 = \frac{1}{2}u_2 + \frac{7}{2}u_3 \\ u_2, u_3 \text{ arbitrary, so let } u_2 = 2r, u_3 = 2s \\ \Rightarrow u_1 = r + 7s \end{array}$$

$$\text{So } \vec{u} = \begin{pmatrix} r+7s \\ 2r \\ 2s \end{pmatrix} = r\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

Basis of eigenspace $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \right\}$

$\Rightarrow$ 2 solutions, one for each basis vector:

$$\vec{x}_1 = e^{5t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = e^{5t}\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

$\lambda = 9$ Solve $(A - 9I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -28 & 12 & 84 \\ 0 & -4 & 0 \\ -8 & 4 & 24 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -28 & 12 & 84 \\ 0 & -4 & 0 \\ -8 & 4 & 24 \end{pmatrix} \xrightarrow[r_3 \to \frac{1}{4}r_3]{r_1 \to \frac{1}{4}r_1} \begin{pmatrix} -7 & 3 & 21 \\ 0 & -4 & 0 \\ -2 & 1 & 6 \end{pmatrix} \xrightarrow{r_3 \to r_3 - \frac{2}{7}r_1}$$

$$\begin{pmatrix} -7 & 3 & 21 \\ 0 & -4 & 0 \\ 0 & \frac{1}{7} & 0 \end{pmatrix} \xrightarrow{r_3 \to r_3 + \frac{7}{4}r_2} \begin{pmatrix} -7 & 3 & 21 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow -7v_1 + 3v_2 + 21v_3 = 0 \Rightarrow v_1 = \frac{3}{7}v_2 + 3v_3$

$-4v_2 = 0 \Rightarrow v_2 = 0 \Rightarrow v_1 = 3v_3$

v_3 arbitrary, so let $v_3 = 1 \Rightarrow v_1 = 3 \Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$$\text{So } \vec{x}_3 = e^{9t}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

General solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

$$\Rightarrow \boxed{\vec{x} = c_1 e^{5t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} + c_3 e^{9t}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}$$