

Modeling with Linear Systems of Equations Using Linear Algebra Techniques

CURM Background Material, Fall 2014
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1 Introduction to Systems of Equations

We can easily solve a small system of equations by hand either directly or by forming the augmented matrix and using Gaussian or Gauss-Jordan elimination. This works well for systems having two to four (or even five) equations with “nice” coefficients. However, for larger systems and/or systems with arbitrary real-valued coefficients, it is less convenient to solve by hand. Maple makes things easier.

There are various ways to solve a system of equations in Maple, including:

- (1) use the *solve* command, or
- (2) form the augmented matrix for the system of equations and transform it to reduced row echelon form using the command *ReducedRowEchelonForm*, found in the *LinearAlgebra* library. (So, we must first include the *LinearAlgebra* library, using the *with* command.) We can then read off the solution.

Unlike some computer algebra systems, when using the *solve* command, Maple can handle systems of equations which have infinitely many solutions or no solution, in addition to problems with a unique solution. Maple will output the solution if there is a unique solution **or** if there are infinitely many solutions, Maple will output a solution in terms of any arbitrary variable(s). In examples to follow, we will look at both approaches.

2 Models with Unique Solutions Using Systems of Equations

2.1 Example 1: Electrical Networks

The electrical network shown in Figure 1 contains batteries (voltage sources) and resistors (e.g., light bulbs or heaters) joined together by conductors (wires). A voltage source provides an electromotive force (emf) E , measured in volts, which moves electrons through the

network. The rate at which electrons flow along a conductor is the current I , measured in amperes (amps). The resistors act to retard the flow of electrons, using up the current (in the form of heat) and lowering the voltage.

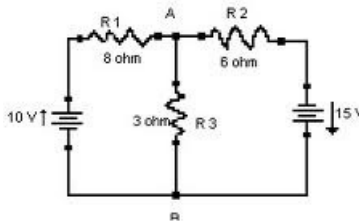


Figure 1: An electrical network.

By Ohm’s law, the voltage drop across a resistor is given by $E = IR$, where I is the current measured in amps and R is the resistance measured in ohms (Ω).

Network analysis is based on Kirchhoff’s laws, which are the following:

Conservation of Energy: Around a closed voltage loop in the network, the algebraic sum of potential differences is zero (voltage drops must balance).

Conservation of Charge: At each node, the total inflow of current equals the total outflow of current.

For our example, the nodes are labeled A and B in Figure 1. Our goal is to determine the values for the currents in each branch of the network. Denote as I_1 the current flowing through R_1 , I_2 the current flowing through R_2 , and I_3 the current flowing through R_3 . Applying Kirchhoff’s laws gives the following system of equations:

$$\begin{aligned} 8I_1 & & - & 3I_3 & = & 10 \\ & & 6I_2 & + & 3I_3 & = & 15 . \\ I_1 & - & I_2 & + & I_3 & = & 0 \end{aligned} \tag{1}$$

Solving (1) using Maple gives $I_1 = \frac{3}{2}$ amps, $I_2 = \frac{13}{6}$ amps, and $I_3 = \frac{2}{3}$ amps.

2.2 Example 2: Fitting a Power Curve – Height vs. Weight

This is a re-do of an example done in the Model Fitting segment of our work. The following table shows the ideal weights for medium built males. The data is copied from *An Introduction to the Mathematics of Biology, with Computer Algebra Models* by Yeargers, Shonkwiler, and Herod.

Height (in)	62	63	64	65	66	67	68	69	70	71	72	73	74
Weight (lb)	128	131	135	139	142	146	150	154	158	162	167	172	177

We would like to fit to this data a curve of the form $y = ax^3 + bx^2 + cx + d$. The method of least squares tells us that we want to

$$\text{Minimize } E = \sum (y_i - (ax_i^3 + bx_i^2 + cx_i + d))^2.$$

As discussed in the Model Fitting notes, in order to do this, we take the partial derivative of E with respect to each of a, b, c, d and set it to 0. So, we obtain the system of equations:

$$\begin{aligned} 0 &= \frac{\partial E}{\partial a} = -2 \sum y_i x_i^3 + 2a \sum x_i^6 + 2b \sum x_i^5 + 2c \sum x_i^4 + 2d \sum x_i^3 \\ 0 &= \frac{\partial E}{\partial b} = -2 \sum y_i x_i^2 + 2a \sum x_i^5 + 2b \sum x_i^4 + 2c \sum x_i^3 + 2d \sum x_i^2 \\ 0 &= \frac{\partial E}{\partial c} = -2 \sum y_i x_i + 2a \sum x_i^4 + 2b \sum x_i^3 + 2c \sum x_i^2 + 2d \sum x_i \\ 0 &= \frac{\partial E}{\partial d} = -2 \sum y_i + 2a \sum x_i^3 + 2b \sum x_i^2 + 2c \sum x_i + 2d, \end{aligned}$$

or

$$\begin{aligned} a \sum x_i^6 + b \sum x_i^5 + c \sum x_i^4 + d \sum x_i^3 &= \sum y_i x_i^3 \\ a \sum x_i^5 + b \sum x_i^4 + c \sum x_i^3 + d \sum x_i^2 &= \sum y_i x_i^2 \\ a \sum x_i^4 + b \sum x_i^3 + c \sum x_i^2 + d \sum x_i &= \sum y_i x_i \\ a \sum x_i^3 + b \sum x_i^2 + c \sum x_i + d \sum 1 &= \sum y_i \end{aligned}$$

Plugging in the values for the sums gives the following system:

$$\begin{aligned} 1343964152934a + 19474949624b + 283011846c + 4124744d &= 632459042 \\ 632459042a + 283011846b + 4124744c + 60294d &= 9195356 \\ 283011846a + 4124744b + 60294c + 884d &= 134084 \\ 4124744a + 60294b + 884c + 13d &= 1961. \end{aligned} \tag{2}$$

The solution to this system is

$$a = \frac{19}{3432}, b = -\frac{163}{154}, c = \frac{1707059}{24024}, d = -\frac{3060027}{2002}.$$

Plotting the data and the polynomial fit on the same graph shows that the solution is visually quite good. (See Figure 2).

3 Models with Infinite Solutions Using Systems of Equations

Here, we will discuss some examples of models whose systems of equations result in infinitely many solutions.

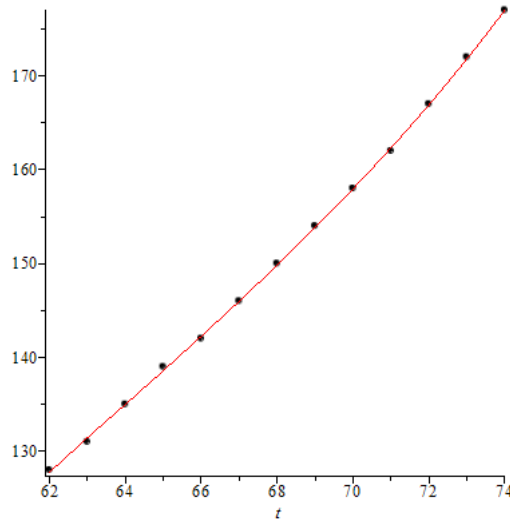


Figure 2: The data and the cubic polynomial fit for the weight vs. height.

3.1 The Leontief Input-Output Model

The following is taken from *Linear Algebra and its Applications*, 4th edition, by David Lay. Leontief’s input-output model is based on the following. Suppose that a nation’s economy is divided into n sectors that produce goods or services, and suppose that $\mathbf{x} \in \mathbb{R}^n$ is a production vector that lists the output of each sector for one year. Also, suppose that there is another sector of the economy, the open sector, that does not produce goods or services but only consumes them, and let \mathbf{d} be a final demand vector that lists the values of the goods and services demanded from the various sectors by the nonproductive sector. The vector \mathbf{d} can represent consumer demand, government consumption, surplus production, exports, or other external demands.

As the various sectors produce goods to meet consumer demand, the producers themselves create additional intermediate demand for goods they need as inputs for their products. The interrelations between the sectors are very complex, and the connection between the final demand and production is unclear. Leontief asked if there is a production level such that the amounts produced will exactly balance the total demanded for that production so that

$$\text{amount produced } \mathbf{x} = \text{intermediate demand} + \text{final demand } \mathbf{d} \quad (3)$$

The basic assumption in Leontief’s input-output model is that for each sector, there is a unit consumption vector in \mathbb{R}^n that lists the inputs needed per unit of output of the sector. All input and output units are measured in millions of dollars, rather than in quantities such as tons or bushels, and prices of goods and services are held constant.

We will look at a simpler version of this, Leontief’s “exchange” model. Suppose a nation’s economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or “exchanged” among the other

sectors of the economy. Let the total dollar value of a sector's output be called the price of that output. Leontief proved that there exist equilibrium prices for the total outputs of the various sectors that result in the income of each sector balancing its expenses (i.e., with no loss or gain in income).

For example, suppose that an economy consists of the coal, electric (power), and steel sectors, and that the output of each sector is distributed among the various sectors as shown in Table 3.1.

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
0.1	0.4	0.6	Coal
0.5	0.1	0.2	Electric
0.4	0.5	0.2	Steel

Table 1: Simple Economy Model

The first column of this table says that the total output of the Coal sector is divided with 10% going to Coal, 50% going to Electric, and 40% going to Steel. The Coal sector would treat the 10% it spends as an operating expense. Notice that all columns sum to 1, because all outputs need to be considered in the table. For this economy, determine the equilibrium prices that make each sector's income match its expenditures. Model using the five-step modeling process.

Step 1: Identify the Problem.

Determine the equilibrium prices for the Coal, Electric, and Steel sectors of the economy so that each sector's income balances its expenditures.

Step 2: Identify Relevant Facts about the Problem.

The relevant information is given to us in the problem.

Step 3: Choose the Type of Modeling Method.

We will be using Leontief's "exchange" model, a deterministic model.

Step 4: Make Simplifying Assumptions.

- Assumptions
 - Leontief's "exchange" model is followed.
 - There are no other sectors in this economy.
 - There are no other consumers of the products.
- Variables
 - p_C = output of Coal sector (million \$/year)

- p_E = output of Electric sector (million \$/year)
- p_S = output of Steel sector (million \$/year)

Step 5: Construct the Model.

Using the data in the table, we construct the following system of equations.

$$\begin{aligned} p_C &= 0.1p_C + 0.4p_E + 0.6p_S \\ p_E &= 0.5p_C + 0.1p_E + 0.2p_S \\ p_S &= 0.4p_C + 0.5p_E + 0.2p_S, \end{aligned}$$

or

$$\begin{aligned} 0.9p_C - 0.4p_E - 0.6p_S &= 0 \\ -0.5p_C + 0.9p_E - 0.2p_S &= 0 \\ -0.4p_C - 0.5p_E + 0.8p_S &= 0 \end{aligned}$$

Step 6: Solve and Interpret the Model.

We solve the system of equations using Maple and find that there are infinitely many solutions, with p_S being the free variable, so

$$p_C = 1.016p_S, p_E = 0.787p_S.$$

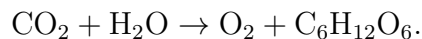
Any nonnegative value for p_S will give a choice of equilibrium prices. For example, choose $p_S = 100$. Then, $p_C = 101.6$ and $p_E = 78.7$. In other words, the Coal sector's price is \$101.6 million, the Electric sector's price is \$78.7 million, and the Steel sector's price is \$100 million.

Note that for this problem, using the *ReducedRowEchelonForm* function in Maple does not give the correct solution due to roundoff error.

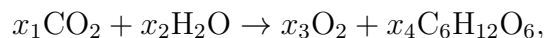
We will omit Steps 7-8 for this problem.

3.2 Balancing Chemical Equations

The following is taken from *Linear Algebra with Applications*, 8th edition, by Steve Leon. In the process of photosynthesis, plants use radiant energy from sunlight to convert carbon dioxide (CO_2) and water (H_2O) into glucose ($\text{C}_6\text{H}_{12}\text{O}_6$) and oxygen (O_2). The chemical equation for this reaction has the form



This equation needs to be balanced. In other words, we must find coefficients x_1, x_2, x_3, x_4 of the reactants and products (i.e., rewrite the equation as



where the numbers of carbon, hydrogen, and oxygen atoms are the same on both sides of the equation). Since carbon dioxide contains one carbon atom and glucose contains six, we require

$$x_1 = 6x_4.$$

Similarly, to balance the oxygen and the hydrogen we need

$$2x_1 + x_2 = 2x_3 + 6x_4 \text{ and } 2x_2 = 12x_4.$$

Moving all of the unknowns to the left-hand sides of the equations gives the homogeneous system of equations

$$\begin{array}{rccccrcr} x_1 & & & & - & 6x_4 & = & 0 \\ 2x_1 & + & x_2 & - & 2x_3 & - & 6x_4 & = & 0 \\ & & 2x_2 & & & - & 12x_4 & = & 0. \end{array}$$

From linear algebra, we know that this system will have infinitely many nontrivial solutions (which is good; otherwise, photosynthesis would be impossible, and we know it is not). From Maple, we see that x_4 is arbitrary, and $x_1 = x_2 = x_3 = 6x_4$. When balancing a chemical equation, we need to use the lowest positive integer solution possible. So, we let $x_4 = 1$, giving $x_1 = x_2 = x_3 = 6$, and the equation takes the form

