Math 90 Practice Midterm III Solutions
Ch. 8-10 (Ebersole), 3.3-3.8 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $f(x) = 3x^4 - 2x + 1$, then $f''(x) = 12x^3 - 2$. **T** $\boxed{\textbf{F}}$

   $f'(x) = 12x^3 - 2$, so $f''(x) = 36x^2$.

2. If $g(x) = (5x^2 + 1)(4x - 2)$, then $g'(x) = (10x)(4)$. **T** $\boxed{\textbf{F}}$

   Using the Product Rule, $g'(x) = (5x^2 + 1)(4) + (4x - 2)(10x) = 20x^2 + 4 + 40x^2 - 20x = 60x^2 - 20x + 4$.

3. $\sec\theta\tan\theta = \frac{\sin\theta}{\cos^2\theta}$ for all angles θ. $\boxed{\textbf{T}}$ **F**

   Notice that $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$. So $\sec\theta\tan\theta = \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$.

4. $\sin(5t) = 5\sin t$ for all angles t. **T** $\boxed{\textbf{F}}$

   This is almost *never* true! For example, if you plug in $t = \frac{\pi}{2}$, on the left hand side you get $\sin\left(\frac{5\pi}{2}\right) = 1$, whereas on the right hand side you get $5\sin\left(\frac{\pi}{2}\right) = 5 \cdot 1 = 5$. So they are not the same!

5. $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$. $\boxed{\textbf{T}}$ **F**

   The reference angle for $\frac{2\pi}{3}$ (the angle in quadrant I corresponding to $\frac{2\pi}{3}$) is $\frac{\pi}{3}$. Using our special triangle we get $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. Since $\frac{2\pi}{3}$ is in quadrant II, and the tangent function is always negative in quadrant II, we get $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$.

6. The only solution to the equation $\cos t = -1$ is $t = \pi$. **T** $\boxed{\textbf{F}}$

   Any angle that is coterminal with π also satisfies the equation. For example, $t = -\pi$, $t = \pm 3\pi$, $t = \pm 5\pi$, etc.

7. $\frac{d}{dt}(\cos(3t^2 + 1)) = -\sin(6t)$. **T** $\boxed{\textbf{F}}$

   Using the Chain Rule, we get $\frac{d}{dt}(\cos(3t^2 + 1)) = -\sin(3t^2 + 1) \cdot 6t = -6t\sin(3t^2 + 1)$.

8. If $3x^2y = \tan(y^2)$, then $\frac{dy}{dx} = \frac{-6xy}{3x^2 - 2y\sec^2(y^2)}$. $\boxed{\textbf{T}}$ **F**

   Taking the derivative of both sides of the equation $3x^2y = \tan(y^2)$ with respect to x, we get

   $$3x^2 \cdot \frac{dy}{dx} + y \cdot 6x = \sec^2(y^2) \cdot 2y \cdot \frac{dy}{dx}.$$

To decide if the statement is true or false we must solve for $\frac{dy}{dx}$:

$$\begin{aligned} 3x^2 \cdot \tfrac{dy}{dx} - 2y\sec^2(y^2) \cdot \tfrac{dy}{dx} &= -6xy \\ (3x^2 - 2y\sec^2(y^2))\tfrac{dy}{dx} &= -6xy \\ \tfrac{dy}{dx} &= \tfrac{-6xy}{3x^2 - 2y\sec^2(y^2)}. \end{aligned}$$

Multiple Choice. *Circle the letter of the best answer.*

1. If $f(x) = \tan x$, then $f''(x) =$

   (a) $\boxed{\frac{2\sin x}{\cos^3 x}}$

   (b) $\frac{1}{\sin^2 x}$

   (c) $\sec^2 x$

   (d) $2\sec x \tan x$

   We have $f'(x) = \sec^2 x$. To get the second derivative we must rewrite $f'(x)$ as $f'(x) = (\sec x)^2$. Now we can use the Chain Rule to get $f''(x) = 2\sec x \cdot \sec x \tan x$. Since none of the answer choices looks like that, we will need to simplify. We get

$$\begin{aligned} f''(x) &= 2\sec x \cdot \sec x \tan x \\ &= 2\sec^2 x \tan x \\ &= 2 \cdot \frac{1}{\cos^2 x} \cdot \frac{\sin x}{\cos x} \\ &= \frac{2\sin x}{\cos^3 x}. \end{aligned}$$

2. $\dfrac{d}{dx}\left(\dfrac{x}{x-1}\right) =$

   (a) $-\frac{x}{(x-1)^2}$

   (b) 1

   (c) $\boxed{-\frac{1}{(x-1)^2}}$

   (d) $\frac{1}{(x-1)^2}$

   Since we cannot simplify $\frac{x}{x-1}$, we need the Quotient Rule in order to take the derivative. We get

$$\frac{d}{dx}\left(\frac{x}{x-1}\right) = \frac{(x-1)(1) - x(1)}{(x-1)^2}. \qquad \text{Simplifying, we get}$$
$$= \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2}.$$

3. If $x^2 - y^2 = 4$, then $\frac{dy}{dx} =$

   (a) $\frac{y}{x}$

   (b) $-\frac{y}{x}$

   (c) $\boxed{\dfrac{x}{y}}$

   (d) $-\frac{x}{y}$

   Here is another implicit differentiation problem. Similar to **True or False** #8, we take the derivative of both sides and solve for $\frac{dy}{dx}$. We have

$$\begin{aligned} 2x - 2y \cdot \frac{dy}{dx} &= 0 \\ 2y \cdot \frac{dy}{dx} &= 2x \\ \frac{dy}{dx} = \frac{2x}{2y} &= \frac{x}{y}. \end{aligned}$$

4. If $f(x) = \sqrt[5]{x^2+1}$, then $f'(x) =$

   (a) $\frac{1}{5}(x^2+1)^{-4/5}$

   (b) $\boxed{\dfrac{2x}{5(x^2+1)^{4/5}}}$

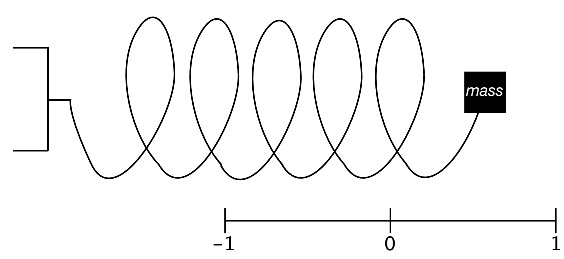
   (c) $\dfrac{2x}{5(x^2+1)^{2/5}}$

   (d) $\frac{1}{5}x(x^2+1)^{-2/5}$

   First rewrite $f(x) = (x^2+1)^{1/5}$. Then use the Chain Rule to get

$$f'(x) = \frac{1}{5}(x^2+1)^{-4/5} \cdot 2x = \frac{2x}{5(x^2+1)^{4/5}}.$$

5. A mass attached to the end of a spring is pulled and then released. t seconds after release, the distance of the mass from equilibrium is $s(t) = \cos 2\pi t$ centimeters. The acceleration of the mass after 3 seconds is



   (a) 0 cm/s^2

   (b) $-4\pi \text{ cm/s}^2$

   (c) $\boxed{-4\pi^2 \text{ cm/s}^2}$

   (d) $-2\pi \text{ cm/s}^2$

Recall that the *acceleration* of a particle (or mass) is the *second derivative* of the distance function. We have

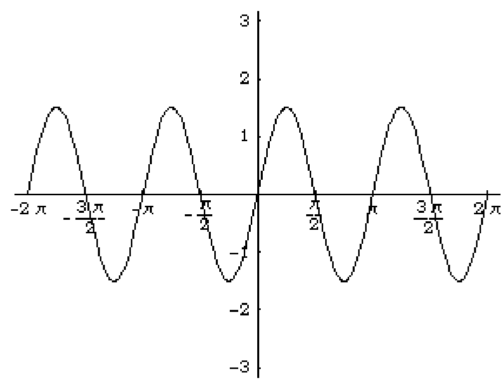
$$\begin{aligned} \text{velocity} &= s'(t) = -\sin 2\pi t \cdot 2\pi = -2\pi \sin 2\pi t \\ \text{acceleration} &= s''(t) = -2\pi \cos 2\pi t \cdot 2\pi = -4\pi^2 \cos 2\pi t. \end{aligned}$$

Plugging in $t = 3$ we get $s''(3) = -4\pi^2 \cos 6\pi = -4\pi^2 \cdot 1 = -4\pi^2$. The units are cm/s per second, or cm/s^2.

Graph.

On the axes at right, sketch a graph of the function $f(t) = \frac{3}{2} \sin 2t$.

More accuracy = more points!



Fill-In.

1. $\sin\left(\frac{3\pi}{2}\right) =$ $\underline{\quad -1 \quad}$

2. $\cos\left(\frac{3\pi}{4}\right) =$ $\underline{\quad -\frac{\sqrt{2}}{2} \quad}$

   The reference angle is $\frac{\pi}{4}$. $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. $\frac{3\pi}{4}$ is in quadrant II, so $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

3. $\tan\left(\frac{11\pi}{6}\right) =$ $\underline{\quad -\frac{\sqrt{3}}{3} \quad}$

   The reference angle is $\frac{\pi}{6}$. $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. $\frac{11\pi}{6}$ is in quadrant IV, so $\tan\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{3}$.

4. $\sec\left(\frac{17\pi}{3}\right) =$ $\underline{\quad 2 \quad}$

   The reference angle is $\frac{\pi}{3}$. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\sec\left(\frac{\pi}{3}\right) = 2$. $\frac{17\pi}{3}$ is in quadrant IV, so $\sec\left(\frac{17\pi}{3}\right) = 2$.

5. If $\cos\theta = -\frac{1}{5}$ and θ is in quadrant II, then

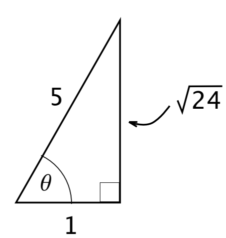
   (a) $\sin\theta =$ $\underline{\quad \frac{\sqrt{24}}{5} \quad}$

   (b) $\tan\theta =$ $\underline{\quad -\sqrt{24} \quad}$

   (c) $\sec\theta =$ $\underline{\quad -5 \quad}$

   (d) $\csc\theta =$ $\underline{\quad \frac{5}{\sqrt{24}} \quad}$

   (e) $\cot\theta =$ $\underline{\quad -\frac{1}{\sqrt{24}} \quad}$



We can "pretend" that θ is in quadrant I and draw a triangle as shown. Notice that the cosine of this "pretend" angle (reference angle) is $\frac{1}{5}$. We can get the third side ($\sqrt{24}$) using the Pythagorean Theorem. Then we find all the trigonometric functions for the reference angle. Finally, since the "real" θ is in quadrant II, we put minus signs on the tangent, secant, and cotangent functions.

6. If $f(-3) = 4$, $f'(-3) = 1$, $f'(2) = 5$, $g(-3) = 2$, and $g'(-3) = -1$, then

   (a) $(f \circ g)'(-3) = \underline{\quad -5 \quad}$

   (b) $(fg)'(-3) = \underline{\quad -2 \quad}$

   (a) Using the Chain Rule, we have $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. Therefore

$$\begin{aligned}(f \circ g)'(-3) &= f'(g(-3)) \cdot g'(-3) \\ &= f'(2) \cdot g'(-3) \\ &= 5 \cdot -1 = -5.\end{aligned}$$

   (b) Using the Product Rule, we have $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$. Therefore

$$\begin{aligned}(fg)'(-3) &= f(-3)g'(-3) + g(-3)f'(-3) \\ &= 4 \cdot -1 + 2 \cdot 1 \\ &= -2.\end{aligned}$$

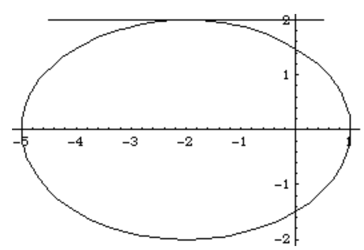
Work and Answer. *You must show all relevant work to receive full credit.*

1. If $f(x) = 4x^{1/3} - \cos x + \frac{1}{\sqrt{x}}$, find $f'(x)$.

   First we rewrite $f(x)$ to make taking the derivative easier: $f(x) = 4x^{1/3} - \cos x + x^{-1/2}$. Then $f'(x) = \boxed{\frac{4}{3}x^{-2/3} + \sin x - \frac{1}{2}x^{-3/2}}$.

2. Find the slope of the tangent line to the graph of $\frac{(x+2)^2}{9} + \frac{y^2}{4} = 1$ at the point $(-2, 2)$.

   To get the slope, we need to get $\frac{dy}{dx}$. One "quick and dirty" way to do it is to notice that the equation represents an ellipse, as shown, and therefore the slope at the point $(-2, 2)$ is 0.



   If you don't think of that, you have to find it by implicit differentiation. This will be easier if we rewrite the equation first:

$$\tfrac{1}{9}(x+2)^2 + \tfrac{1}{4}y^2 = 1.$$

   Then we get

$$\tfrac{2}{9}(x+2) + \tfrac{1}{2}y \cdot \tfrac{dy}{dx} = 0.$$

Now we plug in $x = -2$ and $y = 2$:

$$\tfrac{2}{9}(-2+2) + \tfrac{1}{2}(2) \cdot \tfrac{dy}{dx} = 0$$
$$0 + \tfrac{dy}{dx} = 0.$$

Sure enough, $\frac{dy}{dx} = 0$.

3. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, then its height (in meters) after t seconds is $s(t) = 10t - 0.83t^2$.

   (a) What is the velocity of the stone after 3 seconds?

   To get the velocity, we take the derivative of the distance: $s'(t) = 10 - 1.66t$. Then $s'(3) = 10 - 1.66(3) = \boxed{5.02 \text{ m/s}}$.

   (b) What is the acceleration of the stone after 3 seconds?

   To get the velocity, we take the *second* derivative of the distance, which is the same as the *first* derivative of the velocity: $s''(t) = -1.66$. Then $s''(3) = \boxed{-1.66 \text{ m/s}^2}$.

   (c) When does the stone reach its maximum height?

   When the stone is at its maximum height, the velocity is 0. So we may get the answer by setting the velocity equal to 0 and solving for t:

$$10 - 1.66t \overset{\text{set}}{=} 0$$
$$1.66t = 10$$
$$t = \frac{10}{1.66} = \frac{1000}{166} = \boxed{\frac{500}{83} \approx 6 \text{ seconds}}.$$

4. Find the <u>equation</u> of the tangent line to the graph of $h(x) = \sqrt{5x^2 + 4}$ at $x = 3$.

To get the equation of the line, we first need the slope. So take the derivative: first rewrite $h(x) = (5x^2 + 4)^{1/2}$. Then using the Chain Rule, $h'(x) = \frac{1}{2}(5x^2 + 4)^{-1/2}(10x) = \frac{5x}{\sqrt{5x^2+4}}$. We want the slope at $x = 3$, so plug in 3:

$$h'(3) = \frac{5 \cdot 3}{\sqrt{5 \cdot 3^2 + 4}} = \frac{15}{\sqrt{49}} = \frac{15}{7}.$$

Now we need the y-coordinate of the point of tangency: $h(3) = \sqrt{5 \cdot 3^2 + 4} = \sqrt{49} = 7$. So we have

$$7 = \frac{15}{7} \cdot 3 + b$$
$$7 = \frac{45}{7} + b$$
$$b = 7 - \frac{45}{7} = \frac{49}{7} - \frac{45}{7} = \frac{4}{7}.$$

Therefore the equation is $\boxed{y = \frac{15}{7}x + \frac{4}{7}}$.