Finding the $n$-th term of a sequence - Section 11.1

To develop a formula for the $n$-th term of a sequence $\{a_n\}$, look for the following patterns:

(a) **The Arithmetic Pattern.**

If the terms count up by the same number each time, then the $n$-th term will look like that number times $n$, plus or minus some correction term.

**Example.** $\{-4, -1, 2, 5, 8, 11, \ldots\}$.

Notice that each term in the sequence is 3 more than the one before. Therefore the $n$-th term is $3n - 7$, since $a_1$ must be $-4$. If the first term is $a_0$, then the $n$-th term is $a_n = 3n - 4$, since $a_0$ must be $-4$.

The following are all correct representations of the above sequence:

- $\{3n - 7\}_{n=1}^{\infty}$
- $\{3n - 4\}_{n=0}^{\infty}$
- $\{3n + 2\}_{n=-2}^{\infty}$

(b) **The Power Pattern.**

If the terms are obtained by taking powers of successive integers, then the $n$-th term will look like $(n \pm \xi)^k$, where the $\xi$ depends on where the sequence starts.

**Example.** $\{4, 9, 16, 25, \ldots\}$.

Notice that each term in the sequence is a perfect square, starting with $2^2$. Therefore the $n$-th term is $(n \pm \xi)^2$ for some $\xi$. If the first term is $a_2$, then the $n$-th term is simply $a_n = n^2$, since $a_2$ must be 4. If the first term is $a_1$, then the $n$-th term is $a_n = (n + 1)^2$, since $k a_1$ must be 4.

The following are all correct representations of the above sequence:

- $\{n^2\}_{n=2}^{\infty}$
- $\{(n + 1)^2\}_{n=1}^{\infty}$
- $\{(n + 2)^2\}_{n=0}^{\infty}$

(c) **The Geometric (Exponential) Pattern.**

If the terms are obtained by **multiplying** by the same number each time, then the $n$-th term will look like that number to the $(n \pm \xi)$-th power, where the $\xi$ depends on where the sequence starts.

**Example.** $\{\frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$.

Notice that each term in the sequence is twice the one before. Therefore the $n$-th term is $2^{n \pm \xi}$ for some $\xi$. If the first term is $a_{-1}$, then the $n$-th term is simply $a_n = 2^n$, since $a_{-1}$ must be $\frac{1}{2}$. If the first term is $a_1$, then the $n$-th term is $a_n = 2^{n-2}$, since $a_1$ must be $\frac{1}{2}$.

The following are all correct representations of the above sequence:

- $\{2^{n-2}\}_{n=1}^{\infty}$
- $\{2^{n+1}\}_{n=-2}^{\infty}$
- $\{2^{n}\}_{n=-1}^{\infty}$

(d) **The Alternating Pattern.**

If the terms **alternate**, i.e. switch from positive to negative every term, then the pattern contains a multiple of $(-1)^{n \pm \xi}$ for some $\xi$. If $n \pm \xi$ is even, then the $n$-th term will be positive, so adjust $\xi$ so that the correct terms are positive.

**Example.** $\{\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, \ldots\}$.

The following are all correct representations of the above sequence:

- $\{(-1)^{n-1}\frac{2}{3}\}_{n=1}^{\infty}$
- $\{(-1)^{n+1}\frac{2}{3}\}_{n=1}^{\infty}$
- $\{(-1)^{n}\frac{2}{3}\}_{n=0}^{\infty}$
(e) **A Combination of Patterns.**

Combine the above techniques, but be careful to adjust everything so that the patterns all start correctly.

**Example.** \( \{0, \frac{2}{3}, -\frac{4}{9}, \frac{6}{27}, -\frac{8}{81}, \ldots \} \).

Observe the following patterns:
(i) Counting by 2’s in the numerators
(ii) Powers of 3 in the denominators
(iii) Alternating terms, starting with a negative
Therefore our \(n\)-th term will look something like

\[
(-1)^n \frac{2n \pm \sqrt{3}}{3^n}.
\]

Choose an \(n\) to begin with. I’ll pick \(n = 0\). Then adjust everything so that plugging in \(n = 0\) gives \(a_0 = 0\):

\[
a_n = (-1)^{n+1} \frac{2n}{3^n}.
\]

Alternate solution: start with \(n = 1\). Then the sequence is

\[
\left\{ (-1)^n \frac{2n - 2}{3^{n-1}} \right\}_{n=1}^{\infty}
\]

Notice the following convenient trick:

**Convenient Trick.** To start the sequence from \(n = 1\) instead of \(n = 0\), I replaced all the \(n\)’s in the \(n\)-th term formula by \(n - 1\). Similarly, if I had wanted to begin with \(n = -43\), I could have replaced all the \(n\)’s in the \(n\)-th term formula by \(n + 43\):

\[
\left\{ (-1)^{n+44} \frac{2n + 86}{3^{n+43}} \right\}_{n=-43}^{\infty}
\]

**Practice Problems.**

For each problem, find a formula for \(a_n\). If your first term is not \(a_1\), be sure to make it clear what your first term is by writing the sequence in the form \(\{a_n\}_{n=?}\).

1. \(\{1 \cdot 4, 4 \cdot 8, 7 \cdot 16, 10 \cdot 32, \ldots \}\)

2. \(\{-8, -1, 0, 1, 8, 27, 64, \ldots \}\)

3. \(\{-\frac{1}{6}, \frac{4}{7}, -2, \frac{64}{9}, -\frac{256}{10}, \ldots \}\)