Math 75A Practice Midterm II - Solutions
Ch. 4, 5, 15 (Ebersole), 1.6-2.6 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. The inverse of the function $f(x) = 3x^4 + 1$ is

   (a) $f^{-1}(x) = \sqrt[4]{\dfrac{x-1}{3}}$

   (b) $f^{-1}(x) = \sqrt[4]{\dfrac{x}{3}} - 1$

   (c) $f^{-1}(x) = \sqrt[4]{\dfrac{x}{3} - 1}$

   (d) $\boxed{\text{none of these; } f(x) \text{ does not have an inverse}}$

   $f(x)$ is a polynomial of degree 4; therefore it is not one-to-one and does not have an inverse.

2. If $\ln(3x-2) = 4$, then $x =$

   (a) $\boxed{\dfrac{e^4+2}{3}}$

   (b) $\dfrac{\ln 4 + 2}{3}$

   (c) $\dfrac{4 + \ln 2}{\ln 3}$

   (d) $e^{4/3} + 2$

   $\ln(3x-2) = 4$ means $e^4 = 3x - 2$. Solving for x we get

$$\begin{aligned} 3x &= e^4 + 2 \\ x &= \frac{e^4+2}{3}. \end{aligned}$$

3. If the distance of a train from a station at time t minutes is $s(t) = 30 - t^2$ meters, then the average velocity of the train during the second minute is

   (a) 6 meters per minute

   (b) $\boxed{\text{3 meters per minute}}$

   (c) 4 meters per minute

   (d) 26 meters per minute

   To find the average velocity (the average rate of change of the distance function $s(t)$ during the second minute (from $t = 1$ to $t = 2$) we find the slope of the secant line to the graph through the two points. First we need to find the two points:

$$\begin{aligned} s(1) &= 30 - 1^2 = 29, \text{ so one of the points is } (1, 29). \\ s(2) &= 30 - 2^2 = 26, \text{ so the other point is } (2, 26). \end{aligned}$$

   Therefore the average velocity is

$$\frac{29-26}{1-2} = \frac{3}{-1} = -3 \text{ meters per minute.}$$

The negative answer means the train is getting closer to the station at a rate of $\boxed{\text{3 meters per minute}}$

4. The slope of the tangent line to the graph of $f(x) = |x+2|$ at $x = -3$ is

(a) 1
(b) $\boxed{-1}$
(c) 0
(d) undefined.

$f(x) = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0. \end{cases}$ In other words, $f(x)$ has slope 1 if $x+2 > 0$ and -1 if $x+2 < 0$.
Since $x = -3$ satisfies the second case $(-3+2 < 0)$, the slope there is -1.

5. $\underline{\underline{\text{At } x = 1}}$ the graph of $f(x) = \dfrac{x-1}{x^2 - 4x + 3}$

(a) is continuous
(b) $\boxed{\text{has a hole}}$
(c) has a vertical asymptote
(d) has none of the above

Factor $f(x)$ to get $f(x) = \dfrac{x-1}{(x-1)(x-3)}$. Since the factor $x-1$ cancels, we know there is a hole at $x = 1$. In other words, the graph of $f(x)$ is identical to the graph of $\dfrac{1}{x-3}$ (what we get after cancelling), except at $x = 1$.

6. $\underline{\underline{\text{At } x = 3}}$ the graph of $f(x) = \dfrac{x-1}{x^2 - 4x + 3}$

(a) is continuous
(b) has a hole
(c) $\boxed{\text{has a vertical asymptote}}$
(d) has none of the above

This is the same function as in Multiple Choice #5. Since the factor $x-3$ does not cancel, we know there is a vertical asymptote at $x = 3$.

7. $\underline{\underline{\text{At } x = -2}}$ the graph of $f(x) = \dfrac{|x+2|}{x+2}$

(a) is a horizontal line at $y = 1$
(b) has a hole
(c) has a vertical asymptote
(d) $\boxed{\text{has none of the above}}$

We have $f(x) = \begin{cases} \dfrac{x+2}{x+2} & \text{if } x+2 > 0 \\ -\dfrac{x+2}{x+2} & \text{if } x+2 < 0 \end{cases}$, which simplifies to $\begin{cases} 1 & \text{if } x > -2 \\ -1 & \text{if } x < -2 \end{cases}$. Therefore, as x approaches -2 from the right the graph is at $y = 1$, but as x approaches -2 from the left the graph is at $y = -1$. So there is a break ("jump discontinuity") at $x = -2$.

8. $\underline{\underline{\text{At } x = -1}}$ the graph of $f(x) = \dfrac{|x+2|}{x+2}$

(a) $\boxed{\text{is a horizontal line at } y = 1}$
(b) has a hole
(c) has a vertical asymptote
(d) has none of the above

This is the same function as in Multiple Choice #7. So at $x = -1$ the graph looks like a horizontal line at $y = 1$.

9. For $f(x) = 4x^5 - \pi x^3 + \dfrac{x}{\sqrt{6}}$ and $g(x) = 5x^3 - \dfrac{4}{x} + 2$, which are polynomial functions?

(a) $\boxed{f(x) \text{ only}}$
(b) $g(x)$ only
(c) both $f(x)$ and $g(x)$
(d) neither $f(x)$ nor $g(x)$

We have $f(x) = 4x^5 - \pi x^3 + \dfrac{1}{\sqrt{6}}x$, so $f(x)$ is a polynomial function. On the other hand, $g(x)$ has a term $\dfrac{4}{x}$ with x in the denominator, so $g(x)$ is not a polynomial function.

10. The zeros of the function $f(t) = 5t^2 + 13t - 6$ are

(a) 5 and 13
(b) $\boxed{\frac{2}{5} \text{ and } -3}$
(c) -2 and $\frac{3}{2}$
(d) $\frac{1}{2}$ and -6

$f(t) = (5t - 2)(t + 3)$. Setting $f(t)$ equal to 0, we get $t = \frac{2}{5}$ and $t = -3$.

11. The function $s(t) = \dfrac{t^2 - 9}{t + 3}$

(a) is continuous at $t = -3$
(b) $\boxed{\text{is not continuous at } t = -3.}$

$t = -3$ is not in the domain of $s(t)$. So $s(t)$ cannot be continuous at $t = -3$.

12. Suppose $f(x)$ is a function such that $\lim_{x \to 1} f(x) = 2$. Which of the following is *always* true of $f(x)$?

(a) $f(x)$ is continuous at $x = 1$
(b) $f(1) = 2$
(c) $f(x)$ is continuous on the intervals $(0, 1)$ and $(1, 2)$
(d) $\boxed{\text{None of these.}}$

In order for $f(x)$ to be continuous at $x = 1$, it would have to be true that $\lim_{x \to 1} f(x) = f(1)$. But we are not given any information about whether $f(1)$ is defined or what it might be equal to. So (a) is not always true. And (b) is not always true, for the same reason.

We are also not given any information about whether $f(x)$ is defined except right near $x = 1$. So we have no guarantee that there isn't a problem at, say, $x = \frac{1}{2}$. So (c) does not always hold, since it is possible to have a vertical asymptote, for example, at $x = \frac{1}{2}$ and still have $\lim_{x \to 1} f(x) = 2$.

13. $\lim_{x \to \infty} \dfrac{3x^4 - 4x^2 + 2x - 1}{5 - x^4} =$

(a) $\frac{3}{5}$

(b) $\boxed{-3}$

(c) ∞

(d) 0

The degree of the numerator is equal to the degree of the denominator, so the limit at infinity is equal to the leading coefficient of the top over the leading coefficient of the bottom, or $\dfrac{3}{-1} = -3$. We can also use the "trick" of multiplying the top and bottom by 1 over the biggest power of x in the denominator; we get

$$\begin{aligned} \lim_{x \to \infty} \frac{3x^4 - 4x^2 + 2x - 1}{5 - x^4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} &= \lim_{x \to \infty} \frac{3 - \frac{4}{x^2} + \frac{2}{x^3} - \frac{1}{x^4}}{\frac{5}{x^4} - 1} \\ &= \frac{3}{-1} = -3. \end{aligned}$$

14. If $x < 0$, then $\sqrt[6]{\dfrac{1}{x^{18}}} =$

(a) $\dfrac{1}{x^3}$

(b) $\boxed{-\dfrac{1}{x^3}}$

(c) $\dfrac{1}{x^{1/3}}$

(d) $-\dfrac{1}{x^{1/3}}$

This is the "really awful truth" about $x < 0$. If we plug in $x = -1$, we see that $\sqrt[6]{\dfrac{1}{x^{18}}} \neq \dfrac{1}{x^3}$. But $\sqrt[6]{\dfrac{1}{x^{18}}} = -\dfrac{1}{x^3}$ does hold for $x < 0$.

Fill-In.

1. $\log_2 \left(\dfrac{1}{32} \right) = \underline{\ -5\ }$

$\dfrac{1}{32} = 2^{-5}$.

2. $\log(100) = \underline{\ 2\ }$

$100 = 10^2$.

3. $e^{4\ln 10} =$ <u>10000</u>

$e^{4\ln 10} = e^{\ln 10^4} = 10^4 = 10000.$

4. $\displaystyle\lim_{x\to -1^+} \frac{x-5}{x+1} =$ <u>$-\infty$</u>

$f(x) = \dfrac{x-5}{x+1}$ has a vertical asymptote at $x=-1$, so the limit as x approaches -1 from the right is either ∞ or $-\infty$. Plugging in $x=-0.99$ (a number just to the right of -1), we see that $f(x)$ comes out negative. So the answer to the limit is $-\infty$.

5. $\displaystyle\lim_{x\to 5} \frac{x-5}{x^2-12x+35} =$ <u>$-\frac{1}{2}$</u>

$$\lim_{x\to 5} \frac{x-5}{x^2-12x+35} = \lim_{x\to 5} \frac{x-5}{(x-5)(x-7)} = \lim_{x\to 5} \frac{1}{x-7} = -\frac{1}{2}$$

6. $\displaystyle\lim_{x\to -\infty} \frac{3x^6-4x^5+x^2+3x}{\sqrt{5}x^4-x^3-1} =$ <u>∞</u>

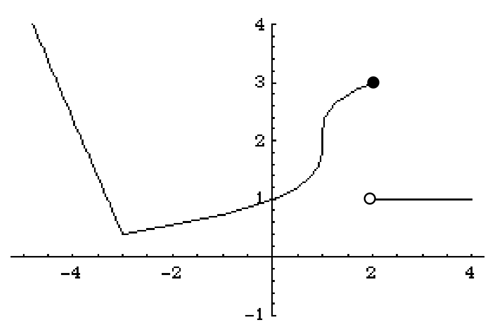
Since the degree of the top is bigger than the degree of the bottom, we know the answer will be either ∞ or $-\infty$. The biggest power of x in the denominator is x^4. So we have

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x^6-4x^5+x^2+3x}{\sqrt{5}x^4-x^3-1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} &= \lim_{x\to -\infty} \frac{3x^2-4x+\frac{1}{x^2}+\frac{3}{x^3}}{\sqrt{5}-\frac{1}{x}-\frac{1}{x^4}} \\ &= \lim_{x\to -\infty} \frac{3x^2-4x}{\sqrt{5}} = \infty.\end{aligned}$$

7. $\displaystyle\lim_{x\to \infty} \frac{2x+3}{4x^3-x+8} =$ <u>0</u>

Since the degree of the bottom is bigger than the degree of the top, we know the answer is 0. You can also get this answer by multiplying the top and bottom by $\frac{1}{x^3}$.

8. Use the graph of $g(t)$ shown at right to answer parts (a) and (b). For each question, list **all** the t-values or largest intervals that make the sentence true.



(a) The value(s) of t at which $g(t)$ is not continuous is/are <u>$t=2$</u> .

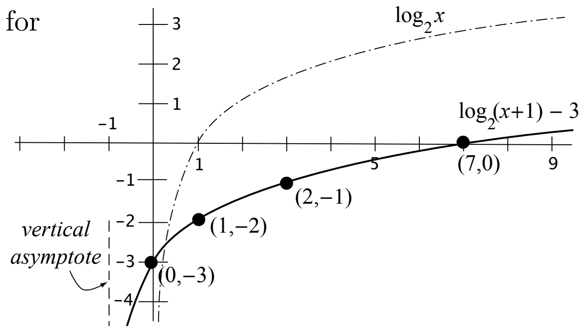
(b) The interval(s) on which $g(t)$ is continuous is/are <u>$(-\infty, 2]$ and $(2,\infty)$</u> .

We use a square bracket in the first interval, since $g(t)$ is continuous from the left at $t=2$.

Graphs. *More accuracy = more points!*

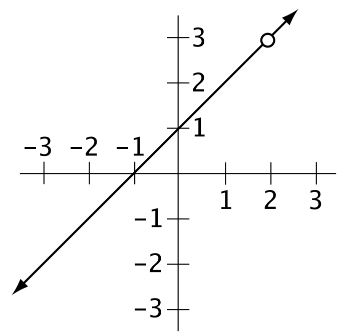
1. On the axes below, sketch a graph of $f(x) = \log_2(x+1) - 3$.
   Label at least two points on the curve.

   The graph is shown along with the graph of $\log_2 x$ for reference.



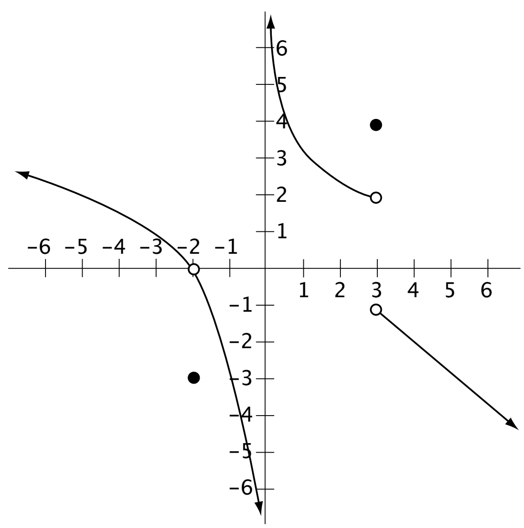
2. On the axes below, sketch a graph of $g(t) = \dfrac{t^2 - t - 2}{t - 2}$.

   Since $\dfrac{t^2 - t - 2}{t - 2} = \dfrac{(t+1)(t-2)}{t-2} = t + 1$ as long as $t \neq 2$, the graph of $g(t)$ is identical to the graph of $t + 1$ except that there is a hole at $t = 2$. Therefore the graph is as shown.



3. On the axes at right, sketch a graph of any function $f(x)$ satisfying all of the following:
   - $\lim\limits_{x \to 3^-} f(x) = 2$
   - $\lim\limits_{x \to 3^+} f(x) = -1$
   - $f(3) = 4$
   - $\lim\limits_{x \to -2} f(x) = 0$
   - $\lim\limits_{x \to 0} f(x)$ does not exist

   One possible graph is shown. There are many correct solutions.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Simplify the expression $\log_3 \left(\dfrac{27x^4}{3^{y+2}} \right)$.

   We have $\log_3 \left(\dfrac{27x^4}{3^{y+2}} \right) = \log_3(27) + \log_3\left(x^4\right) - \log_3\left(3^{y+2}\right) = 3 + 4\log_3 x - (y+2) = \boxed{4\log_3 x - y + 1}$

2. Compute $\lim_{x\to 0^-} \dfrac{|x|-x}{x}$. If the limit does not exist, explain why.

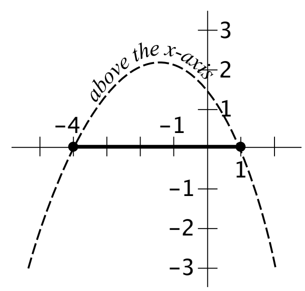
$$\frac{|x|-x}{x} = \begin{cases} \dfrac{x-x}{x} & \text{if } x>0 \\ \dfrac{-x-x}{x} & \text{if } x<0 \end{cases} \quad \text{(we write } > \text{ instead of } \geq \text{ since } \tfrac{|x|-x}{x} \text{ is undefined at } x=0\text{)}$$
$$= \begin{cases} \dfrac{0}{x} & \text{if } x>0 \\ \dfrac{-2x}{x} & \text{if } x<0 \end{cases}$$
$$= \begin{cases} 0 & \text{if } x>0 \\ -2 & \text{if } x<0. \end{cases}$$

Since we are only interested in the limit *from the left*, we need only consider the second case above. So $\lim_{x\to 0^-} \dfrac{|x|-x}{x} = \boxed{-2}$.

3. Find the domain of the function $f(x) = \sqrt[20]{-x^2-3x+4}$. *Express your answer in interval notation.*

We must have $-x^2-3x+4 \geq 0$ since it is under an even root (the 20th root). First we find the zeros of $-x^2-3x+4$ by setting it equal to 0; we get

$$\begin{aligned} -x^2-3x+4 &= 0 \\ -(x^2+3x-4) &= 0 \\ -(x+4)(x-1) &= 0. \end{aligned}$$



So the zeros are $x=-4$ and $x=1$. Since the graph of $-x^2-3x+4$ is a parabola opening down, we know that the positive part is in between the two zeros (see graph at right). Therefore the domain of $f(x)$ is $\boxed{[-4,1]}$

4. Compute $\lim_{x\to\infty} \dfrac{|x|-x}{x}$. If the limit does not exist, explain why.

This is similar to Work and Answer #2, except here we are taking the limit as x approaches ∞. Since the function has output 0 for all $x>0$ (see the above solution to #2), we conclude that $\lim_{x\to\infty} \dfrac{|x|-x}{x} = \boxed{0}$

5. For the function $f(x) = \dfrac{2x^2-x-3}{x^2-4x-5}$,

(a) Find the equation(s) of the vertical asymptote(s) of $f(x)$.

(b) Find the equation(s) of the horizontal asymptote(s) of $f(x)$.

(a) Factoring, we get $f(x) = \dfrac{(2x-3)(x+1)}{(x-5)(x+1)}$. $f(x)$ is undefined at $x=5$ and $x=-1$, since those are the zeros of the denominator. But $x+1$ cancels, so there is no vertical asymptote at $x=-1$. Therefore the only vertical asymptote is at $\boxed{x=5}$

(b) To find horizontal asymptotes, we take the limits at infinity; we have

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2 - x - 3}{x^2 - 4x - 5} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} &= \lim_{x\to\infty} \frac{2 - \frac{1}{x} - \frac{3}{x^2}}{1 - \frac{4}{x} - \frac{5}{x^2}} \\ &= \frac{2}{1} = 2.\end{aligned}$$

The limit of $f(x)$ as x approaches $-\infty$ is exactly the same; therefore the only horizontal asymptote is $\boxed{y = 2}$

6. Compute $\displaystyle\lim_{x\to\infty} \frac{\sqrt[4]{7x^{12} - 4x^3 + 5}}{2x - \sqrt{5}x^3}$. If the limit does not exist, explain why.

We have $\displaystyle\frac{1}{x^3} = \sqrt[4]{\frac{1}{x^{12}}}$ for $x > 0$, so

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt[4]{7x^{12} - 4x^3 + 5}}{2x - \sqrt{5}x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} &= \lim_{x\to\infty} \frac{\sqrt[4]{7x^{12} - 4x^3 + 5}\,\sqrt[4]{\frac{1}{x^{12}}}}{\frac{2}{x^2} - \sqrt{5}} \\ &= \lim_{x\to\infty} \frac{\sqrt[4]{7 - \frac{4}{x^9} + \frac{5}{x^{12}}}}{\frac{2}{x^2} - \sqrt{5}} \\ &= \boxed{\frac{\sqrt[4]{7}}{\sqrt{5}}}\end{aligned}$$

7. Compute $\displaystyle\lim_{x\to-\infty} \frac{\sqrt[4]{7x^{12} - 4x^3 + 5}}{2x - \sqrt{5}x^3}$. If the limit does not exist, explain why.

This is similar to Work and Answer #6, above, except that here we are taking the limit of the function as x approaches $-\infty$. Since the function has a fourth root, the "really awful truth" applies. We have $\displaystyle\frac{1}{x^3} = -\sqrt[4]{\frac{1}{x^{12}}}$ for $x < 0$, so the answer comes out the **opposite** to that of #6. Repeat the steps above, but put in an extra minus sign, and you should get

$$\lim_{x\to-\infty} \frac{\sqrt[4]{7x^{12} - 4x^3 + 5}}{2x - \sqrt{5}x^3} = \boxed{-\frac{\sqrt[4]{7}}{\sqrt{5}}}$$