**Math 76 Practice Problems for Midterm III - Solutions**
Chapter 11

**DISCLAIMER.** This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

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**Multiple Choice.** *Circle the letter of the best answer.*

1. The $n$th term of the sequence $\{-3, 4, 11, 18, 25, \ldots\}$, counting $a_1 = -3$ as the first term, is

   (a) $a_n = 5n - 2$

   $\boxed{\text{(b)}}$ $a_n = 7n - 10$

   (c) $a_n = n^2 - 4$

   (d) $a_n = -3n + 7$

   This is an arithmetic sequence with a common difference of 7, so only (b) can be correct. Sure enough, when $n = 1$, $a_1 = 7n - 10 = 7 - 10 = -3$.

2. $\displaystyle\sum_{n=1}^{\infty} 3\left(\tfrac{1}{2}\right)^n =$

   (a) 6

   $\boxed{\text{(b)}}$ 3

   (c) $\dfrac{3}{2}$

   (d) $\infty$ (diverges)

   This is a geometric series with $r = \dfrac{1}{2}$, so it converges. But take care! The sum is $\displaystyle\sum_{n=1}^{\infty} 3\left(\tfrac{1}{2}\right)^n = \frac{3}{1-\frac{1}{2}} - 3 = 6 - 3 = 3$ since the sum starts from $n = 1$, not $n = 0$.

3. The series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}}$

   (a) converges to $\dfrac{8}{9}$

   $\boxed{\text{(b)}}$ converges to $\dfrac{1}{9}$

   (c) converges to 3

   (d) converges to 9

   This is a geometric series. There are several ways to get it into a form that fits the formula. Here are two:

   **Solution 1.**

   We have $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}} = \sum_{n=3}^{\infty} \frac{2}{3^n}.$

This is exactly in the form we want it, but there are three terms "missing." (The formula $\dfrac{a}{1-r}$ works when the series starts from $n=0$, but this one starts at $n=3$.) So we take $\dfrac{a}{1-r}$ and subtract off the terms corresponding to $n=0$, $n=1$, and $n=2$. We get

$$\frac{2}{1-\frac{1}{3}} - 2 - \frac{2}{3} - \frac{2}{9} = 3 - 2 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9}.$$

**Solution 2.**

We have $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{2}{3^{n+3}} = \sum_{n=0}^{\infty} \frac{2}{3^3}\left(\frac{1}{3}\right)^n = \frac{\frac{2}{27}}{1-\frac{1}{3}} = \frac{1}{9}.$

4. $\displaystyle\sum_{n=3}^{\infty} \left(\frac{2}{n} - \frac{2}{n+1}\right) =$

(a) $0$

(b) $\dfrac{1}{6}$

$\boxed{\text{(c)}}$ $\dfrac{2}{3}$

(d) $\infty$ (diverges)

This is a telescoping series. The $n$-th partial sum is

$$s_n = \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{6}\right) + \ldots + \left(\frac{2}{n} - \frac{2}{n+1}\right) = \frac{2}{3} - \frac{2}{n+1},$$

whose limit as $n \to \infty$ is $\dfrac{2}{3}$.

5. To determine whether or not the series $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3}{1-2n+n^4}$ converges, the limit comparison test may be used with comparison series $\sum b_n =$

$\boxed{\text{(a)}}$ $\displaystyle\sum \frac{1}{n}$

(b) $\sum 5n^3$

(c) $\displaystyle\sum \frac{5}{n^4}$

(d) none; the limit comparison test cannot be used

The degree of the denominator of $a_n = \dfrac{5n^3}{1-2n+n^4}$ is one more than the degree of the numerator. So the best comparison term is $b_n = \dfrac{1}{n}$. To check, note that the limit of $\frac{a_n}{b_n}$ is finite and positive, since

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{5n^3}{1-2n+n^4}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{5n^3}{1-2n+n^4} \cdot \frac{n}{1} \\ &= \lim_{n\to\infty} \frac{5n^4}{1-2n+n^4} = 5. \end{aligned}$$

6. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n^2-4\sqrt{n}-1}$

   (a) $\boxed{\text{converges absolutely (AC)}}$
   (b) converges conditionally (CC)
   (c) diverges

   Since the biggest power on the bottom is $n^2$ and the biggest power on the top is $n^{1/2}$, the difference in the powers is greater than 1 ($2-\frac{1}{2}=\frac{3}{2}$). Thus the series converges absolutely (AC) by the limit comparison test, using $b_n = \dfrac{1}{n^{3/2}}$.

   Here is a more detailed solution:

   Try for AC: look at the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2-4\sqrt{n}-1}$. The terms $a_n$ of this series are positive, at least from some point on. So we may use the limit comparison test. Let $b_n = \dfrac{1}{n^{3/2}}$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{n^{1/2}}{n^2-4n^{1/2}-1}\cdot\frac{n^{3/2}}{1}\\ &= \lim_{n\to\infty}\frac{n^2}{n^2-4n^{1/2}-1}=1,\end{aligned}$$

   a finite positive limit. Therefore we are using the right $b_n$ for the limit comparison test. Since $\sum b_n$ converges (it is a $p$-series with $p=\frac{3}{2}$), our series also converges. In other words, the original series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n^2-4\sqrt{n}-1}$ converges *absolutely* (AC).

7. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n-4\sqrt{n}-1}$

   (a) converges absolutely (AC)
   (b) $\boxed{\text{converges conditionally (CC)}}$
   (c) diverges

   Here the difference in the powers is less than 1 ($1-\frac{1}{2}=\frac{1}{2}$). Thus the series will not converge absolutely. However, it will still converge (conditionally), by the alternating series test: we have

   - Let $f(x) = \dfrac{\sqrt{x}}{x-4\sqrt{x}-1}$. Then

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x-4\sqrt{x}-1)-\sqrt{x}(1-\frac{2}{\sqrt{x}})}{(x-4\sqrt{x}-1)^2} = -\frac{\frac{1}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)}{(x-4\sqrt{x}-1)^2}<0$$

     (I skipped a lot of algebra here; you can check my work). Therefore the terms are decreasing.

- $\lim_{n\to\infty} \frac{\sqrt{n}}{n - 4\sqrt{n} - 1} = 0$ since the power on the bottom is bigger than the power on the top.

8. The series $\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{7n!}$

   (a) $\boxed{\text{converges absolutely (AC)}}$

   (b) converges conditionally (CC)

   (c) diverges

   Using the ratio test we get

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{10^{n+1}}{7(n+1)!} \cdot \frac{7n!}{10^n} \\ &= \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1. \end{aligned}$$

   Therefore the series is AC.

9. The series $\sum_{n=2}^{\infty} \left( \frac{2n^2+1}{n^2+5n-6} \right)^n$

   (a) converges absolutely (AC)

   (b) converges conditionally (CC)

   (c) $\boxed{\text{diverges}}$

   Using the root test we get

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left( \frac{2n^2+1}{n^2+5n-6} \right)^n} \\ &= \lim_{n\to\infty} \frac{2n^2+1}{n^2+5n-6} = 2 > 1. \end{aligned}$$

   Therefore the series diverges.

10. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n$ is

   (a) $[0, 1]$

   (b) $(0, 1)$

   (c) $(0, 2]$

   $\boxed{\text{(d)}}$ $[0, 2)$

   Since the power series is centered at $x = 1$, we can see immediately that the answer must be either (c) or (d) (You can also check this using the ratio test).

Now we check the endpoints: at $x = 0$ we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ which converges by the alternating series test. At $x = 2$ we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges by the $p$-series test.

11. A power series representation for the function $f(x) = \dfrac{3}{4-x}$ is

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{3}{4}x^n$

$\boxed{\text{(c)}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{3}{4^{n+1}}x^n$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+1}x^n$

(d) $\displaystyle\sum_{n=0}^{\infty}3(4-x)^n$

We have

$$\frac{3}{4-x} = \frac{3}{4\left(1-\frac{1}{4}x\right)} = \frac{\frac{3}{4}}{1-\frac{1}{4}x}$$
$$= \sum_{n=0}^{\infty}\frac{3}{4}\left(\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{3}{4^{n+1}}x^n.$$

12. The Maclaurin series for the function $f(x) = x^3\cos(4x^2)$ is

$\boxed{\text{(a)}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-16)^n}{(2n)!}x^{4n+3}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(4x^2)^n$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n+3}$

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{4n^2+3}$

Recall that the Maclaurin series for $\cos(x)$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$, for all $x$. Therefore for $\cos(4x^2)$ it is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(4x^2)^{2n}$. Finally, for $x^3\cos(4x^2)$ it is

$$x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(4x^2)^{2n} = x^3\sum_{n=0}^{\infty}\frac{(-1)^n4^{2n}x^{4n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-16)^n}{(2n)!}x^{4n+3}.$$

13. The Maclaurin series for the function $f(x) = \sqrt{3-x}$ is

(a) $\displaystyle\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{3^n}x^n$

(c) $\displaystyle\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$

$\boxed{\text{(b)}}$ $\displaystyle\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\sqrt{3}\left(-\frac{1}{3}\right)^n x^n$

(d) $\displaystyle\sum_{n=0}^{\infty}\binom{n}{\frac{1}{2}}\frac{(-1)^n}{3^n}x^n$

We have

$$\sqrt{3-x} = \sqrt{3}\sqrt{1-\frac{x}{3}} = \sqrt{3}\left(1-\frac{x}{3}\right)^{1/2}$$
$$= \sqrt{3}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\sqrt{3}\left(-\frac{1}{3}\right)^n x^n.$$

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**Fill-In.**

1. *Circle the best answer. On the line, indicate one valid test that can be applied to get your answer. You may choose from the following list:*

- divergence test
- $p$-series test
- geometric series test
- integral test
- direct comparison test
- limit comparison test
- alternating series test
- ratio test
- root test

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{500}{n^2}$ ( $\boxed{\textbf{converges}}$ | **diverges** )

Test: $p$-series test, integral test, limit comparison test

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n-5}{3n}$ ( **converges** | $\boxed{\textbf{diverges}}$ ) Test: divergence test

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{3\sqrt{n}}{n^2-3n+1}$ ( $\boxed{\textbf{converges}}$ | **diverges** )

Test: limit comparison test, integral test

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\tan^{-1}(n)}$ ( **converges** | $\boxed{\textbf{diverges}}$ ) Test: divergence test

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{(5n)!}$ ( $\boxed{\textbf{converges}}$ | **diverges** ) Test: ratio test

2. $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n^2+1}(x+3)^{n-1}$ is a power series centered at $-3$ .

A power series of the form $\displaystyle\sum_{n=?}^{\infty}c_n(x-a)^n$ is centered at $a$. Here we have $a=-3$.

3. The radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}x^n$ is 1 .

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}|x|$$
$$= |x| \overset{\text{set}}{<} 1.$$

Therefore the radius of convergence is $R = 1$.

4. A power series representation of $6\tan^{-1}(x^2)$ is $\underline{\displaystyle\sum_{n=0}^{\infty} \frac{6(-1)^n}{2n+1} x^{4n+2}}$.

There are three ways to solve this problem:

(a) **Differentiate and integrate.**
The derivative of $6\tan^{-1}(x^2)$ is $\dfrac{12x}{1+x^4} = \displaystyle\sum_{n=0}^{\infty} 12x(-x^4)^n = \sum_{n=0}^{\infty} 12(-1)^n x^{4n+1}$; thus $6\tan^{-1}(x^2) = \displaystyle\sum_{n=0}^{\infty} \frac{12(-1)^n}{4n+2} x^{4n+2} + C = \sum_{n=0}^{\infty} \frac{6(-1)^n}{2n+1} x^{4n+2} + C$. Finally, plugging in $x = 0$, we see that $C = 0$.

(b) **Modify the power series for $\tan^{-1}(x)$.**
If you have memorized the Maclaurin series expansion for $\tan^{-1}(x)$, you can plug in $x^2$ for $x$ and multiply the whole thing by 6. We have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

so

$$6\tan^{-1}(x^2) = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{6(-1)^n}{2n+1} x^{4n+2}.$$

(c) **Find the Maclaurin series by hand** (difficult)**.**
Take successive derivatives of $f(x) = 6\tan^{-1}(x^2)$ and try to get a formula for $f^{(n)}(0)$. This takes a huge amount of time and is not recommended!

5. The Maclaurin series of $\cos(3x)$ is $\underline{\displaystyle\sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!} x^{2n}}$.

The Maclaurin series of $\cos x$ (which you should memorize) is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. Therefore the Maclaurin series of $\cos(3x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (3x)^{2n} = \sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!} x^{2n}$.

6. The Taylor series of $\cos(3x)$ centered at $\pi$ is $\underline{\displaystyle -\sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!} (x-\pi)^{2n}}$.

We don't have any Taylor series not centered at 0 memorized (at least I don't!), so this one we have to do from scratch. We have

| $n$ | $f^{(k)}(x)$ | $f^{(k)}(\pi)$ |
|---|---|---|
| 0 | $\cos(3x)$ | $-1$ |
| 1 | $-3\sin(3x)$ | 0 |
| 2 | $-3^2\cos(3x)$ | $3^2$ |
| 3 | $3^3\sin(3x)$ | 0 |
| 4 | $3^4\cos(3x)$ | $-3^4$ |
| 5 | $-3^5\sin(3x)$ | 0 |
| 6 | $-3^6\cos(3x)$ | $3^6$ |

We are starting to see the pattern. When $k$ is odd, $f^{(k)}(\pi) = 0$. When $k$ is even, $f^{(k)}(\pi) = (-1)^{n-1}3^k$. So let $k = 2n$; then we get $\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}3^{2n}}{(2n)!}(x-\pi)^{2n} = -\sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!}(x-\pi)^{2n}$ (notice the extra minus sign to make the simplification come out right. No, you don't have to simplify on the test!).

7. $\binom{7}{3} = \underline{\ 35\ }$.

The seventh row of Pascal's triangle is

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

so $\binom{7}{3} = 35$. Alternatively, the formula for $\binom{7}{3}$ is $\frac{7!}{4!3!} = \frac{7\cdot 6\cdot 5}{3\cdot 2} = 35$.

8. $\binom{\frac{4}{3}}{5} = \underline{\ -\frac{8}{3^6}\ }$.

We have

$$\binom{\frac{4}{3}}{5} = \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\cdots\left(\frac{4}{3}-5+1\right)}{5!} = \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{5!}$$
$$= -\frac{4\cdot 2\cdot 5\cdot 8}{3^5\cdot 5!} = -\frac{8}{3^6}.$$

**Work and Answer.** *You must show all relevant work to receive full credit.*

1. For the sequence $\{2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\}$,

   (a) Find a formula for the $n$-th term $a_n$ of the sequence, assuming $a_0 = 2$.

   We make the following observations:

   (i) The sequence is alternating, so there is a $(-1)^{\xi}$ in the formula.

   (ii) The numerators are proceeding by powers of 2, so there is a $2^{\spadesuit}$ in the numerator of the formula.

(iii) The denominators are proceeding by powers of 3, so there is a $3^{\clubsuit}$ in the denominator of the formula.

We have $a_0 = 2 = +\dfrac{2^1}{3^0}$, so the formula for $a_n$ is $\boxed{a_n = \ (-1)^n \dfrac{2^{n+1}}{3^n}}$

(b) *Circle the best answer.* The sequence $\{a_n\}_{n=0}^{\infty}$ above ( **converges** | **diverges** ).

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2 \cdot \left(\dfrac{2}{3}\right)^n = 0$. So the sequence converges to 0.

2. Find the sum of the series $\displaystyle\sum_{n=-1}^{\infty} \frac{2 \cdot 3^n}{4^{n-1}}$.

This is a geometric series after some manipulation. Note that

$$\begin{aligned}
\sum_{n=-1}^{\infty} \frac{2 \cdot 3^n}{4^{n-1}} &= \sum_{n=-2}^{\infty} \frac{2 \cdot 3^{n+1}}{4^n} = \sum_{n=-2}^{\infty} \frac{6 \cdot 3^n}{4^n} = \sum_{n=-2}^{\infty} 6 \cdot \left(\frac{3}{4}\right)^n \\
&= \frac{6}{1-\frac{3}{4}} + 6 \cdot \left(\frac{3}{4}\right)^{-1} + 6 \cdot \left(\frac{3}{4}\right)^{-2} \\
&= 24 + 6\left(\frac{4}{3}\right) + 6\left(\frac{16}{9}\right) = \boxed{\frac{128}{3}}
\end{aligned}$$

3. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^3 - n}$.

This is a telescoping series after some manipulation. Note that

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n+1}{n^3 - n} &= \sum_{n=2}^{\infty} \frac{n+1}{n(n+1)(n-1)} \\
&= \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right)
\end{aligned}$$

(using partial fractions). Therefore

$$\begin{aligned}
s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 1 - \frac{1}{n}
\end{aligned}$$

which approaches 1 as $n \to \infty$. Therefore the sum of the series is $\boxed{1}$

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{3^n \sin n}{n!}$ is absolutely convergent (AC), conditionally convergent (CC), or divergent.

Try for AC: we want to check whether or not $\sum_{n=1}^{\infty} \left| \frac{3^n \sin n}{n!} \right|$ converges or not. We have

$$\left| \frac{3^n \sin n}{n!} \right| = \frac{3^n |\sin n|}{n!} \leq \frac{3^n}{n!}$$

since $|\sin n| \leq 1$ for all $n$. We are attempting to use the direct comparison test — however, we need another test to determine whether or not $\sum \frac{3^n}{n!}$ converges. You can check using the ratio test that $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ does converge. Therefore $\sum_{n=1}^{\infty} \left| \frac{3^n \sin n}{n!} \right|$ also converges, and hence the original series $\boxed{\textbf{converges absolutely (AC)}}$

5. (a) Find a power series representation for the function $f(x) = \ln(2 + 3x)$.

There are two ways to do this problem. One is to use §11.10 and find, say, the Maclaurin series for $f(x)$. This is quite difficult, however. Here's the way I recommend:

Use §11.9 and recognize that $f'(x) = \dfrac{3}{2+3x} = \dfrac{\frac{3}{2}}{1 - \left(-\frac{3}{2}x\right)}$, which is the sum

$$\sum_{n=0}^{\infty} \frac{3}{2} \left( -\frac{3}{2} x \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{2^{n+1}} x^n,$$

and therefore $f(x)$ is equal to an antiderivative of this sum: we get

$$\ln(2+3x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{(n+1)2^{n+1}} x^{n+1} + C = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} x^n + C.$$

Plugging in $x = 0$ we see that $\ln(2 + 3(0)) = \ln 2 = C$. Therefore we have

$$\ln(2+3x) = \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} x^n}$$

(b) Find the interval of convergence.

Using the ratio test, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{3^n x^n} \right| \\ &= \lim_{n\to\infty} \frac{3n}{2(n+1)} |x| = \frac{3}{2}|x| \overset{\text{set}}{<} 1 \end{aligned}$$

So $|x| < \dfrac{2}{3}$. It remains to check the endpoints.

When $x = \dfrac{2}{3}$ we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} \left(\frac{2}{3}\right)^n$ (the $(\ln 2)$ at the beginning won't affect whether the series converges or not), which equals $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. This **converges** by the alternating series test.

When $x = -\dfrac{2}{3}$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} \left(-\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which **diverges** since it is a $p$-series with $p = 1$.

Therefore the interval of convergence is $\boxed{\left(-\frac{2}{3}, \frac{2}{3}\right]}$

6. (a) Write the Taylor series for the function $f(x) = \sqrt{x}$ centered at 1.

We have

$$\begin{aligned}
f(x) &= \sqrt{x} & f(1) &= 1 \\
f'(x) &= \frac{1}{2\sqrt{x}} & f'(1) &= \frac{1}{2} \\
f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2} & f''(1) &= \frac{1}{2}\left(-\frac{1}{2}\right) \\
f'''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2} & f'''(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \\
f^{(4)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-7/2} & f^{(4)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \quad \ldots
\end{aligned}$$

We can see now the pattern that we get. We have

$$\begin{aligned}
f^{(5)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5} \\
f^{(6)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right) = -\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^6} \\
\ldots \quad f^{(n)}(1) &= (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}.
\end{aligned}$$

By Taylor's Theorem the coefficients of the Taylor series for $f(x)$ are $c_n = \dfrac{f^{(n)}(1)}{n!}$. For $n \geq 2$ this follows the pattern as above, and we have $c_n = (-1)^{n-1} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!\, 2^n}$.

The first two terms (for $n = 0$ and $n = 1$) do not follow this pattern, so we just write them out separately; we get

$$\boxed{\sqrt{x} = 1 + \frac{1}{2}(x-1) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\,2^n}(x-1)^n}$$

(b) Find the radius of convergence.

Using the ratio test we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(x-1)^{n+1}}{(n+1)!\,2^{n+1}}\cdot\frac{n!\,2^n}{1\cdot 3\cdot 5\cdots(2n-3)\cdot(x-1)^n}\right|\\ &= \lim_{n\to\infty}\frac{2n-1}{2(n+1)}|x-1|\\ &= |x-1| \overset{\text{set}}{<} 1.\end{aligned}$$

Therefore the radius of convergence is $\boxed{R = 1}$

(c) Estimate $\sqrt{1.4}$ using the first three terms of the Taylor series.

$\sqrt{1.4} \approx 1 + \frac{1}{2}(0.4) - \frac{1}{2!\cdot 4}(0.4)^2 = 1 + 0.2 - 0.02 = \boxed{1.18}$

(For comparison, a calculator gives $\sqrt{1.4} \approx 1.1832$.)

7. Estimate $\displaystyle\int_0^1 e^{x^2}\,dx$ using the first two terms of the Maclaurin series expansion.

First we have that the Maclaurin series for $e^x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Since the radius of convergence is infinite we can substitute in $x^2$ for $x$ to get the Maclaurin series for $e^{x^2}$, which is $\displaystyle\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$. Integrating term by term we get

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!\,(2n+1)}\Bigg|_0^1\\ &= \left(\sum_{n=0}^{\infty}\frac{1^{2n+1}}{n!\,(2n+1)}\right) - \left(\sum_{n=0}^{\infty}\frac{0^{2n+1}}{n!\,(2n+1)}\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!\,(2n+1)}.\end{aligned}$$

Now just evaluate the first 2 terms to get the approximation:

$$\approx \frac{1}{1} + \frac{1}{3} = \boxed{\frac{4}{3}}$$

8. Estimate $\displaystyle\int_0^1 \sin x^2\, dx$ using the first two terms of the Maclaurin series expansion.

This is similar to the previous problem.

First we have that the Maclaurin series for $\sin x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Since the radius of convergence is infinite we can substitute in $x^2$ for $x$ to get the Maclaurin series for $\sin(x^2)$, which is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$. Integrating term by term we get

$$\begin{aligned}
\int_0^1 \sin x^2\, dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} \Bigg|_0^1 \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1^{4n+3}}{(4n+3)\cdot(2n+1)!}\right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0^{4n+3}}{(4n+3)\cdot(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}.
\end{aligned}$$

Now just evaluate the first 2 terms to get the approximation:

$$\approx \frac{1}{3\cdot 1!} - \frac{1}{7\cdot 3!} = \boxed{\frac{13}{42}}$$

9. (a) Write out and simplify $\binom{\frac{4}{5}}{n}$ for $n \geq 2$.

   (b) Write the binomial series of $(1-2x^3)^{4/5}$ and use (a) to simplify.

   (a) First we have

$$\binom{\frac{4}{5}}{n} = \frac{\left(\frac{4}{5}\right)\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\cdots\left(\frac{4}{5}-n+1\right)}{n!}. \tag{1}$$

The key to simplifying this expression is to observe that $\frac{4}{5} - n + 1 = -\frac{5n-9}{5}$; therefore for $n = 2$ the binomial coefficient would look like $\binom{\frac{4}{5}}{2} = \frac{\left(\frac{4}{5}\right)\left(-\frac{1}{5}\right)}{2!}$, for $n = 3$ it would be $\binom{\frac{4}{5}}{2} = \frac{\left(\frac{4}{5}\right)\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{3!}$, and so on. Now we see that (1) equals

$$\boxed{\frac{(-1)^{n-1} 4\cdot 1\cdot 6\cdot 11\cdots(5n-9)}{5^n\, n!}} \tag{2}$$

   (b) We can use the formula (2) for $n = 2$ and up. For $n = 0$ and $n = 1$ we'll just write those terms out separately. We have $\binom{\frac{4}{5}}{0} = 1$ and $\binom{\frac{4}{5}}{1} = \frac{4}{5}$, so

$$\begin{aligned}
(1-2x^3)^{4/5} &= \sum_{n=0}^{\infty} \binom{\frac{4}{5}}{n} (-2x^3)^n \\
&= 1 - \frac{4}{5} \cdot 2x^3 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 4 \cdot 1 \cdot 6 \cdot 11 \cdots (5n-9)}{5^n \; n!} \cdot (-1)^n 2^n x^{3n} \\
&= \boxed{1 - \frac{8}{5} x^3 - \sum_{n=2}^{\infty} \frac{4 \cdot 1 \cdot 6 \cdot 11 \cdots (5n-9)}{n!} \cdot \left(\frac{2}{5}\right)^n x^{3n}}
\end{aligned}$$

**Remark.** We started from $n = 2$ because $n = 0$ and $n = 1$ do not seem to fit the pattern in (2). However, if we rewrite (2) as

$$\frac{(-1)^n \cdot -4 \cdot 1 \cdot 6 \cdot 11 \cdots (5n-9)}{5^n \; n!}$$

it does work for $n = 1$. Thus, another way to write the binomial series for $(1-2x^3)^{4/5}$ is

$$1 + \sum_{n=1}^{\infty} \frac{-4 \cdot 1 \cdot 6 \cdot 11 \cdots (5n-9)}{n!} \cdot \left(\frac{2}{5}\right)^n x^{3n}.$$