Math 75B Practice Problems for Midterm III – Solutions

§§4.2-5.4

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $f(x)$ is a continuous function and $f(3) = 2$ and $f(5) = -1$, then $f(x)$ has a root between 3 and 5. $\boxed{\textbf{T}}$ **F**

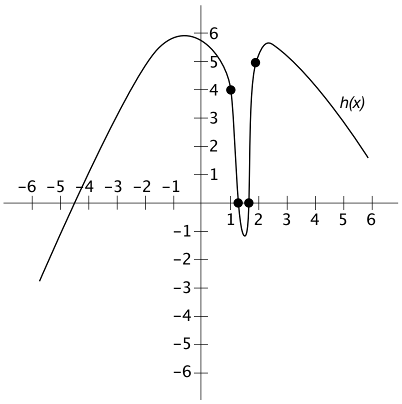
   This is the Intermediate Value Theorem. Since $f(3) > 0$ and $f(5) < 0$, the function must cross the x-axis somewhere between 3 and 5.

2. The function $g(x) = 2x^3 - 12x + 5$ has 5 real roots. **T** $\boxed{\textbf{F}}$

   Since the equation $g'(x) = 6x^2 - 12 = 0$ has 2 solutions, by Rolle's Theorem $g(x)$ has no more than 3 real roots.

3. If $h(x)$ is a continuous function and $h(1) = 4$ and $h(2) = 5$, then $h(x)$ has no roots between 1 and 2. **T** $\boxed{\textbf{F}}$

   As an example, the function at right is continuous and has $h(1) = 4$ and $h(2) = 5$, but there are **two** roots between 1 and 2.



4. The only x-intercept of $f(x) = x^3 - x^2 + 2x - 2$ is $(1, 0)$. *(Challenge problem!)* $\boxed{\textbf{T}}$ **F**

   It is true that $(1, 0)$ is an x-intercept of $f(x)$, since $1^3 - 1^2 + 2(1) - 2 = 0$. We can then use Rolle's Theorem to show that $x = 1$ is the *only* root of $f(x)$. We have $f'(x) = 3x^2 - 2x + 2$. If we set $f'(x) = 0$ and use the quadratic formula, we get

   $$x = \frac{2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{2 \pm \sqrt{-20}}{6},$$

   which are not real numbers. Therefore there is at most one root of $f(x)$. So $x = 1$ must be it.

5. If the velocity of an object at time t is $v(t) = 4t^2 + 1$ ft./s, then its distance in feet at time t is $s(t) = \frac{4}{3}t^3 + t$. **T** $\boxed{\textbf{F}}$

   The distance might be $s(t) = \frac{4}{3}t^3 + t + 450$. There is no initial condition in the problem. Another way to put this is, the problem does not say "distance *from* somewhere." Maybe $s(t)$ represents the distance from Egypt! Since we don't know, we can't say for sure which antiderivative represents the distance.

6. The function $F(x) = \sin 2x + 52$ is an antiderivative of the function $f(x) = 2\cos 2x$. $\boxed{\textbf{T}}$ **F**

We can check this by taking the derivative of $F(x)$. We get $F'(x) = \cos 2x \cdot 2 = 2\cos 2x = f(x)$.

7. The function $G(x) = 4x^3$ is an antiderivative of the function $g(x) = x^4 - 2$. **T** $\boxed{\textbf{F}}$

$G'(x) = 12x^2 \neq g(x)$.

8. $-1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} = \sum_{i=-1}^{4} \frac{i}{i+2}$. $\boxed{\textbf{T}}$ **F**

This is a tricky problem, because the pattern on the left-hand side of the equation is disguised. However, all we have to do is find the sum on the right and see if it simplifies to the sum on the left:

$$\begin{aligned} \sum_{i=-1}^{4} \frac{i}{i+2} &= \frac{-1}{1} + \frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} \\ &= -1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3}. \end{aligned}$$

Sure enough, the equation is valid.

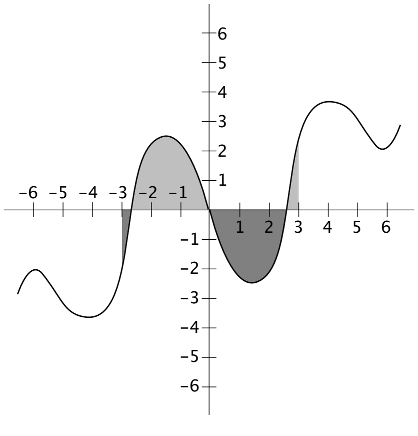
9. $\sum_{i=2}^{4} \frac{i^2}{2} = \frac{29}{2}$. $\boxed{\textbf{T}}$ **F**

We have

$$\sum_{i=2}^{4} \frac{i^2}{2} = \frac{2^2}{2} + \frac{3^2}{2} + \frac{4^2}{2} = \frac{4+9+16}{2} = \frac{29}{2}.$$

10. If $g(x)$ is an odd function which is continuous on the interval $[-3, 3]$, then $\int_{-3}^{3} g(x)\,dx = 0$. $\boxed{\textbf{T}}$ **F**

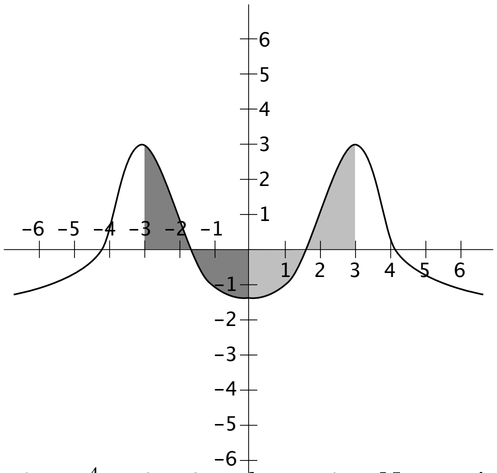
If $g(x)$ is an odd function, that means the graph of $g(x)$ is symmetric about the origin. Therefore between -3 and 3 there is the same amount of area above the x-axis as below. (An example is shown at right.)



11. If $h(x)$ is an even function which is continuous on the interval $[-3, 3]$, then $\int_{-3}^{3} h(x)\,dx = 2\int_{0}^{3} h(x)\,dx$. $\boxed{\textbf{T}}$ **F**

This is also true! If $h(x)$ is an even function, that means the graph of $h(x)$ is symmetric about the y-axis. Therefore whatever area there is between 0 and 3 there is the same amount of area between -3 and 0. (An example is shown at right.) So we can just double the amount we get from 0 to 3.

This can be a very useful fact when doing definite integrals, since 0 is a lot easier to plug in than -3.



Multiple Choice. *Circle the letter of the best answer.*

1. If $x_1 = 1$ is a first approximation of a solution to the equation $x^4 = 6 - 3x$, then using Newton's Method the second approximation is $x_2 =$

$\boxed{\text{(a)}}$ $\frac{9}{7}$

(b) $\frac{5}{7}$

(c) $\frac{9}{2}$

(d) $-\frac{5}{2}$

First we let $f(x) = x^4 + 3x - 6$. Then finding a solution to the equation $x^4 = 6 - 3x$ is the same as finding a root of $f(x)$. The formula for x_2 is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Now $f(1) = 1 + 3 - 6 = -2$ and $f'(x) = 4x^3 + 3$, so $f'(1) = 7$, and we get

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-2}{7} \\ &= 1 + \frac{2}{7} = \boxed{\frac{9}{7}}. \end{aligned}$$

2. $\int_{-2}^{2} \sqrt{4 - x^2}\,dx =$

(a) $-\frac{1}{6}$

(b) 0

$\boxed{\text{(c)}}$ 2π

(d) does not exist.

The curve $y = \sqrt{4 - x^2}$ is the upper half of a circle with radius 2 centered at $(0, 0)$. Therefore the integral $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$ represents the area of a semicircle of radius 2, which is $\frac{1}{2} \cdot \pi \cdot 2^2 = \boxed{2\pi}$.

3. $\int_{0}^{\pi/4} \sec x \tan x\,dx =$

$\boxed{\text{(a)}}$ $\sqrt{2}-1$

(b) $\sqrt{2}$

(c) $1-\frac{\sqrt{2}}{2}$

(d) does not exist.

The function $f(x)=\sec x\tan x$ is defined on the interval $\left[0,\frac{\pi}{4}\right]$, so the integral exists. Using the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_0^{\pi/4}\sec x\tan x\;dx &= \sec x\Big|_0^{\pi/4}\\ &= \sec\left(\frac{\pi}{4}\right)-\sec(0)\\ &= \boxed{\sqrt{2}-1}\end{aligned}$$

4. If $f(x)$ is continuous, then $\displaystyle\int_0^{\pi/4} f(x)\;dx-\int_0^{\pi} f(x)\;dx=$

(a) $\displaystyle\int_{\pi/4}^{\pi} f(x)\;dx$

$\boxed{\text{(b)}}$ $\displaystyle\int_{\pi}^{\pi/4} f(x)\;dx$

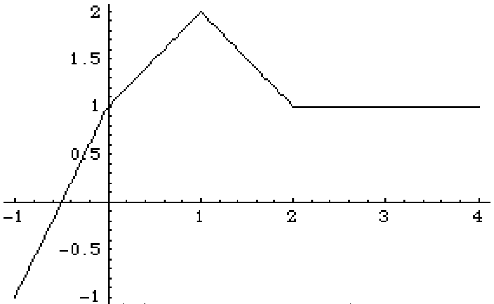
(c) $\displaystyle\int_0^{3\pi/4} f(x)\;dx$

(d) $\displaystyle\int_{\pi}^{0} f(x)\;dx$

This is just the property $\displaystyle\int_a^c f(x)\;dx-\int_a^b f(x)\;dx=\int_b^c f(x)\;dx.$

For #5-6, use the graph of $f(x)$ shown below to answer the questions.

5. $\displaystyle\int_{-1}^{0} f(x)\;dx\;=$



(a) -1

$\boxed{\text{(b)}}$ 0

(c) 1

(d) 2

Between $x=-1$ and $x=0$ there is just as much area (between $f(x)$ and the x-axis) above the x-axis as below. Therefore the net area is $\boxed{0}$.

6. $\displaystyle\int_{2}^{1} f(x)\;dx\;=$

$\boxed{\text{(a)}}$ $-\frac{3}{2}$

(b) $\frac{3}{2}$

(c) -1

(d) 1

The area under $f(x)$ from $x=1$ to $x=2$ is $\frac{3}{2}$. But the limits of integration have been reversed. So

$$\begin{aligned}\int_2^1 f(x)\;dx &= -\int_1^2 f(x)\;dx\\ &= \boxed{-\frac{3}{2}}\end{aligned}$$

Fill-In. *If an answer is undefined, write "D.N.E."*

1. If a polynomial function $f(x)$ has 3 solutions to the equation $f'(x) = 0$, then $f(x)$ has at most $\underline{4}$ roots.

   According to Rolle's Theorem, if a continuous, differentiable function (such as a polynomial function) has n places where the derivative is 0, then there are at most $n + 1$ real roots.

2. $\int (\sqrt[3]{x} - \sec^2 x)\, dx = \underline{\tfrac{3}{4}x^{4/3} - \tan x + C}$.

   $\sqrt[3]{x} = x^{1/3}$, so $\int (\sqrt[3]{x} - \sec^2 x)\, dx = \frac{3}{4}x^{4/3} - \tan x + C$ (this is straight out of the formulas in §5.4, p. 351 of Stewart).

3. $\int_{-1}^{2} \sqrt[4]{x}\, dx = \underline{\text{ D.N.E. }}$

   $\sqrt[4]{x}$ is undefined for $x < 0$. Therefore the integral does not make sense.

4. $\int_{-1}^{2} \sqrt[3]{x}\, dx = \underline{\tfrac{3}{4}\left(2\sqrt[3]{2} - 1\right)}$

   Here we have no domain problems — $\sqrt[3]{x}$ is defined for all x. So we use F.T.C. (Fundamental Theorem of Calculus):

$$\begin{aligned}\int_{-1}^{2} \sqrt[3]{x}\, dx &= \int_{-1}^{2} x^{1/3}\, dx \\ &= \frac{3}{4}x^{4/3}\Big|_{-1}^{2} \\ &= \frac{3}{4}\left(2^{4/3} - (-1)^{4/3}\right) \\ &= \frac{3}{4}\left(2\sqrt[3]{2} - 1\right).\end{aligned}$$

   You can also leave your answer as $\frac{3}{4}\left(2^{4/3} - 1\right)$ or anything equal to it. Simplify whenever you can, but *at your own risk!*

5. If $F(x) = \int_{5}^{x} \sqrt{5t - t^4}\, dt$, then $F'(x) = \underline{\sqrt{5x - x^4}}$.

   This is exactly what F.T.C. Part 1 says.

6. If $G(x) = \int_{x}^{\sqrt{\pi}} \cot(6t^2)\, dt$, then $G'(x) = \underline{-\cot(6x^2)}$.

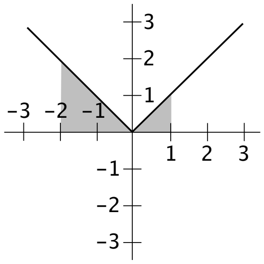
   We have $G(x) = \int_{x}^{\sqrt{\pi}} \cot(6t^2)\, dt = -\int_{\sqrt{\pi}}^{x} \cot(6t^2)\, dt$, so using F.T.C. Part 1 we get the answer shown (the opposite to what we would have gotten if the x had been in the upper limit in the original problem).

7. If the interval $[-4, 7]$ is divided into 6 equal subintervals, then the width of each subinterval is $\underline{\ \frac{11}{6}\ }$.

   The interval $[-4, 7]$ is 11 ($= 7 - (-4)$) units wide. If we divide 11 units into 6 equal parts, each part will be $\frac{11}{6}$ units wide.

8. $\displaystyle\int_{-2}^{1} |x|\, dx = \underline{\ \frac{5}{2}\ }$.

   The easiest way to do this problem is to think about **areas**. The graph of $|x|$, with the area from -2 to 1 shaded, is shown at right.



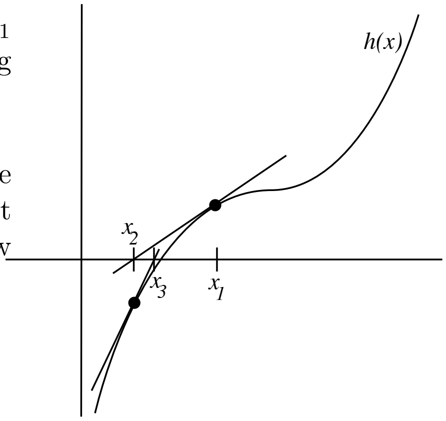
   Notice that we have two triangles. The areas of the triangles are $\frac{2\cdot 2}{2} = 2$ and $\frac{1\cdot 1}{2} = \frac{1}{2}$. Since they are both above the x-axis, we add:

$$\int_{-2}^{1} |x|\, dx = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$$

Graphs.

1. For the function $h(x)$ graphed at right and the initial guess x_1 shown, draw tangent lines to determine x_2 and x_3 according to Newton's Method. Label x_2 and x_3 on the x-axis.

   Newton's Method takes each guess, finds the equation of the tangent line to $f(x)$ at that x-value, and gets the x-intercept of that line, which becomes the next guess. You can see how this is graphed at right.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Prove that the function $f(x) = -x^3 - 6x + 1$ has exactly one real root by completing the following:

   (a) Use the Intermediate Value Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at least* one real root.

   $f(0) = 1 > 0$ and $f(1) = -1 - 6 + 1 < 0$. Since $f(x)$ is continuous on the interval $[0, 1]$, there must be at least one root between 0 and 1.

   (b) Use Rolle's Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at most* one real root.

   Since $f(x)$ is a polynomial function, it is continuous and differentiable everywhere. Therefore Rolle's Theorem applies. We have $f'(x) = -3x^2 - 6 \overset{\text{set}}{=} 0 \Rightarrow x^2 = -2$ has **no solution**. Therefore $f(x)$ cannot have more than **one** real root.

2. Estimate the root of $f(x) = x^3 + 2x - 1$ using two iterations of Newton's Method (*i.e.* compute x_3) with the initial guess $x_1 = 0$. *Express your answer as an exact fraction.*

We use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ twice. $f'(x) = 3x^2 + 2$, so we get

$$\begin{aligned} x_2 &= 0 - \tfrac{f(0)}{f'(0)} \\ &= 0 - \tfrac{-1}{2} = \tfrac{1}{2}\,. \end{aligned} \qquad f(0) = -1 \qquad f'(0) = 2$$

$$\begin{aligned} x_3 &= \tfrac{1}{2} - \tfrac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} \\ &= \tfrac{1}{2} - \tfrac{\frac{1}{8}}{\frac{11}{4}} \\ &= \tfrac{1}{2} - \tfrac{4}{8\cdot 11} \\ &= \tfrac{1}{2} - \tfrac{1}{22} = \tfrac{11-1}{22} = \tfrac{10}{22} = \boxed{\tfrac{5}{11}}\,. \end{aligned} \qquad \begin{aligned} f\left(\tfrac{1}{2}\right) &= \left(\tfrac{1}{2}\right)^3 + 2\left(\tfrac{1}{2}\right) - 1 = \tfrac{1}{8} \\ f'\left(\tfrac{1}{2}\right) &= 3\left(\tfrac{1}{2}\right)^2 + 2 = \tfrac{3}{4} + 2 = \tfrac{11}{4} \end{aligned}$$

3. Evaluate $\displaystyle\int \frac{2}{t^3}\,dt$.

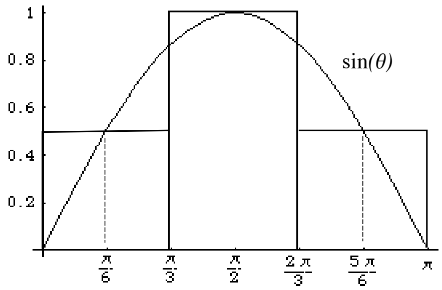
We have

$$\begin{aligned} \int \frac{2}{t^3}\,dt &= \int 2t^{-3}\,dt \\ &= 2 \cdot \frac{1}{-2} t^{-2} + C = -t^{-2} + C = \boxed{-\frac{1}{t^2} + C} \end{aligned}$$

4. Evaluate $\displaystyle\int \frac{2}{1+x^2}\,dx$.

5. (a) <u>Estimate</u> $\displaystyle\int_0^{\pi} \sin\theta\,d\theta$ using 3 rectangles and midpoints.

We are dividing the interval $[0, \pi]$ into 3 equal parts, so each part is of width $\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{3} = \frac{\pi}{3}$. So the three subintervals are $\left[0, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and $\left[\frac{2\pi}{3}, \pi\right]$. The midpoints of these intervals are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$ (see picture at right).



The heights of the three rectangles, therefore, are

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
- $\sin\left(\frac{\pi}{2}\right) = 1$
- $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

Therefore the area of the rectangles, put together, is

$$\begin{aligned} \left(\frac{1}{2} + 1 + \frac{1}{2}\right)\Delta x &= \left(\frac{1}{2} + 1 + \frac{1}{2}\right) \cdot \frac{\pi}{3} \\ &= 2 \cdot \frac{\pi}{3} = \boxed{\frac{2\pi}{3}} \end{aligned}$$

(b) Evaluate $\int_0^\pi \sin\theta \, d\theta$ exactly.

Using FTC-2, we get

$$\begin{aligned}\int_0^\pi \sin\theta \, d\theta &= -\cos\theta \Big|_0^\pi \\ &= (-\cos\pi) - (-\cos 0) = -(-1) + 1 = \boxed{2}\end{aligned}$$

(c) What is the error of the estimate you made in part (a)?

The error is $\frac{2\pi}{3} - 2 \approx 0.094$, a slight overestimate (π is a little more than 3, so $\frac{2\pi}{3}$ is a little more than 2). Looking at the graph and the rectangles, it seems plausible that our estimate would be a little too big, but not by much.

6. If $F(x) = \int_5^{\sin^2 x} (3t - 5) \, dt$,

(a) Evaluate $F'(x)$.

Using the chain rule, we get

$$F'(x) = (3\sin^2 x - 5) \cdot 2\sin x \cos x = \boxed{6\sin^3 x \cos x - 10 \sin x \cos x}$$

(b) Evaluate $F(x)$.

We have

$$\begin{aligned}F(x) = \int_5^{\sin^2 x} (3t - 5) \, dt &= \frac{3}{2}t^2 - 5t \Big|_5^{\sin^2 x} \\ &= \left(\frac{3}{2}(\sin^2 x)^2 - 5\sin^2 x\right) - \left(\frac{3}{2} \cdot 5^2 - 5 \cdot 5\right) \\ &= \boxed{\frac{3}{2}\sin^4 x - 5\sin^2 x - \frac{75}{2} + 25}\end{aligned}$$

(c) Show that the derivative of the function you obtained in (b) equals the function you obtained in (a).

Taking the derivative of the function $F(x)$ found in (a), we get

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{2}\sin^4 x - 5\sin^2 x - \frac{75}{2} + 25\right) &= \frac{3}{2} \cdot 4\sin^3 x \cos x - 10 \sin x \cos x \\ &= 6\sin^3 x \cos x - 10 \sin x \cos x,\end{aligned}$$

which is the answer we got in (a).

7. Evaluate $\int_0^{\pi/3} x^2 - \sin x \, dx$.

The function $f(x) = x^2 - \sin x$ is continuous on the interval $\left[0, \frac{\pi}{3}\right]$, so the integral makes sense. We have

$$\begin{aligned}\int_0^{\pi/3} x^2 - \sin x \; dx &= \frac{1}{3}x^3 + \cos x \,\Bigg|_0^{\pi/3} \\ &= \left(\frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \cos\left(\frac{\pi}{3}\right)\right) - (0+1) \\ &= \boxed{\frac{\pi^3}{81} + \frac{1}{2} - 1}\end{aligned}$$

8. An object travels in a straight line with velocity function $v(t) = \frac{3}{t} - 4e^t$ feet per second. Determine the net change in position (in feet) over the time interval $2 \le t \le 5$.

The net change in position is $s(5) - s(2)$, where $s(t)$ is the position function. By F.T.C. we have

$$\begin{aligned}s(5) - s(2) &= \int_2^5 \frac{3}{t} - 4e^t \; dt \\ &= 3\ln|t| - 4e^t \,\Bigg|_2^5 \\ &= \left(3\ln 5 - 4e^5\right) - \left(3\ln 2 - 4e^2\right) \\ &= 3(\ln 5 - \ln 2) - 4e^5 + 4e^2 \\ &= \boxed{3\ln\left(\tfrac{5}{2}\right) - 4e^2(e^3 - 1)}\end{aligned}$$