Math 75A Practice Midterm I – Solutions
§§2-A – 4-C (Ebersole), 1.1-1.5 (Stewart), **W1**

Multiple Choice. *Circle the letter of the best answer.*

1. A description for the function $f(x) = \sqrt{3x} + 2$ is

   (a) Take 3 times a number and then add 2
   (b) Take 3 times a number, add 2, and then take the square root of the result
   (c) [Take 3 times a number, take the square root of the result, then add 2]
   (d) Take $\sqrt{3}$ times a number and then add 2

   $3x$ is under the square root, so we are taking the input and multiplying it by 3, then taking the square root of the result. Finally, we add 2.

2. The range of the function $g(x) = -x^2 + 6x + 5$ is

   (a) $\mathbb{R}$ (all real numbers)
   (b) $[14, \infty)$
   (c) $[-\infty, 14)$
   (d) $\boxed{(-\infty, 14]}$

   $g(x)$ is a parabola opening down, so the range (set of outputs) must be from $-\infty$ to the y-coordinate of the vertex. The vertex is at $(3, 14)$ (for a reminder of how to find the vertex of a parabola, see p. 65-66 of Ebersole). Therefore the range is $(-\infty, 14]$ since 14 is included but $-\infty$ is not ($-\infty$ is not a real number!).

3. The graph of the function $g(t) = \sqrt{9 - t^2}$ is

   (a) A circle of radius 9 centered at the origin
   (b) A circle of radius 3 centered at the origin
   (c) The upper half of a circle of radius 9 centered at the origin
   (d) [The upper half of a circle of radius 3 centered at the origin]

   $y = \sqrt{r^2 - t^2}$ always represents the upper half of a circle of radius r centered at the origin, since if we square both sides we get $y^2 = r^2 - t^2$, or $t^2 + y^2 = r^2$, which is the equation of a circle of radius r. We get only the upper half because $\sqrt{r^2 - t^2}$ cannot be negative.

4. If the distance of a train from a station at time t minutes is $s(t) = 30 - t^2$ meters, then the average velocity of the train during the second minute is

   (a) 6 meters per minute
   (b) [3 meters per minute]
   (c) 4 meters per minute
   (d) 26 meters per minute

The second minute starts at time $t = 1$ and goes to $t = 2$. The average velocity is always

$$\frac{\text{distance traveled}}{\text{time elapsed}};$$

in this case the train travels $s(2) - s(1)$ meters in 1 second, so we have

$$\frac{s(2) - s(1)}{1} = (30 - 2^2) - (30 - 1^2) = 26 - 29 = -3 \text{ meters per minute.}$$

The answer comes out negative because the distance of the train from the station has actually decreased, i.e. the train is moving *toward* the station at an average speed of 3 meters per minute.

5. At $x = 1$ the graph of $f(x) = \dfrac{x - 1}{x^2 - 4x + 3}$

(a) is continuous

(b) $\boxed{\text{has a hole}}$

(c) has a vertical asymptote

(d) has none of the above

By factoring and cancelling we can see that

$$f(x) = \frac{x - 1}{(x - 1)(x - 3)} = \frac{1}{x - 3} \quad \text{as long as } x \neq 1$$

So the graph of $f(x)$ is identical to that of $\dfrac{1}{x - 3}$, except that there is a hole at $x = 1$.

6. At $x = -2$ the graph of $f(x) = \dfrac{|x + 2|}{x + 2}$

(a) is a horizontal line at $y = 1$

(b) has a hole

(c) has a vertical asymptote

(d) $\boxed{\text{has none of the above}}$

We have

$$f(x) = \begin{cases} \dfrac{x + 2}{x + 2} & \text{if } x + 2 > 0 \\ \\ \dfrac{-(x + 2)}{x + 2} & \text{if } x + 2 < 0 \end{cases}$$
$$= \begin{cases} 1 & \text{if } x > -2 \\ -1 & \text{if } x < -2 \end{cases}$$

Therefore the graph has a break at $x = -2$ (it looks like the horizontal line $y = 1$ for $x > -2$, but looks like $y = -1$ for $x < -2$. The function is undefined at $x = -2$).

7. At $x = -1$ the graph of $f(x) = \dfrac{|x+2|}{x+2}$

(a) $\boxed{\text{is a horizontal line at } y = 1}$

(b) has a hole

(c) has a vertical asymptote

(d) has none of the above

From the above problem, since -1 is a number greater than -2, the graph of $f(x)$ looks like the horizontal line $y = 1$ there.

8. For $f(x) = 4x^5 - \pi x^3 + \dfrac{x}{\sqrt{6}}$ and $g(x) = 5x^3 - \dfrac{4}{x} + 2$, which are polynomial functions?

(a) $\boxed{f(x) \text{ only}}$

(b) $g(x)$ only

(c) both $f(x)$ and $g(x)$

(d) neither $f(x)$ nor $g(x)$

$f(x)$ can be rewritten as $4x^5 - \pi x^3 + \dfrac{1}{\sqrt{6}}x$, so it has only whole-number powers of x. The function $g(x)$ cannot be written only in terms of whole-number powers of x, since $\frac{4}{x} = 4x^{-1}$.

9. The zeros of the function $f(t) = 5t^2 + 13t - 6$ are

(a) 5 and 13

(b) $\boxed{\tfrac{2}{5} \text{ and } -3}$

(c) -2 and $\frac{3}{2}$

(d) $\frac{1}{2}$ and -6

To find the zeros (or *roots*) of a function, we set it equal to zero and solve:

$$\begin{gathered} 5t^2 + 13t - 6 \overset{\text{set}}{=} 0 \\ (5t-2)(t+3) = 0 \\ 5t - 2 = 0 \quad \text{or} \quad t + 3 = 0 \\ 5t = 2 \quad \text{or} \quad t = -3 \\ t = \frac{2}{5}, \quad t = -3. \end{gathered}$$

10. The function $s(t) = \dfrac{t^2 - 9}{t + 3}$

(a) is continuous at $t = -3$

(b) $\boxed{\text{is not continuous at } t = -3.}$

$t = -3$ is not in the domain of $s(t)$. Therefore the function cannot be continuous there.

11. Suppose $f(x)$ is a function such that $\lim_{x\to 1} f(x) = 2$. Which of the following is *always* true of $f(x)$?

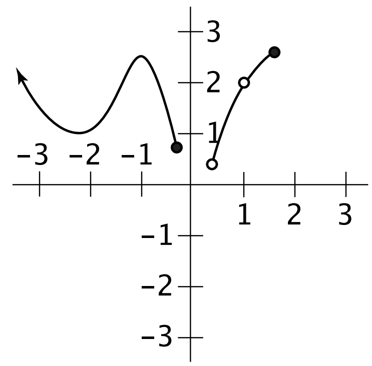
(a) $f(x)$ is continuous at $x = 1$

(b) $f(x)$ is continuous on the intervals $(0, 1)$ and $(1, 2)$

(c) $f(1) = 2$

(d) None of these.

The limit says nothing about the value of $f(x)$ *at* $x = 1$, only what the function is *trying* to do as x gets *close* to 1 (from both sides). We are not even guaranteed that the function is "well-behaved" on the intervals $(0, 1)$ and $(1, 2)$; for instance, the graph at right shows a function $f(x)$ for which $\lim_{x\to 1} f(x) = 2$, but none of the first three answer choices applies.



Fill-In.

1. If $f(x) = 3x - 5$ and $g(x) = x^3$, then

(a) $(g \circ f)(1) =$ <u>-8</u>
(b) $(g - f)(0) =$ <u>5</u>
(c) $(f \circ f)(2) =$ <u>-2</u>
(d) $(f \circ g)(-1) =$ <u>-8</u>

These problems can be done in two ways. You can either just **plug the specific inputs** into each function, or you can **compute the formulas** for $(g\circ f)(x)$, $(g-f)(x)$, $(f\circ f)(x)$, and $(f \circ g)(x)$ and then plug in the values to each new formula.

Method 1.

(a) $f(1) = 3(1) - 5 = -2$, so $(g \circ f)(1) = g(f(1)) = g(-2) = (-2)^3 = -8$.
(b) $(g - f)(0) = g(0) - f(0) = 0^3 - (3(0) - 5) = 0 - (-5) = 5$.
(c) $f(2) = 3(2) - 5 = 1$, so $(f \circ f)(2) = f(f(2)) = f(1) = 3(1) - 5 = -2$.
(d) $g(-1) = (-1)^3 = -1$, so $(f \circ g)(-1) = f(g(-1)) = f(-1) = 3(-1) - 5 = -8$.

Method 2.

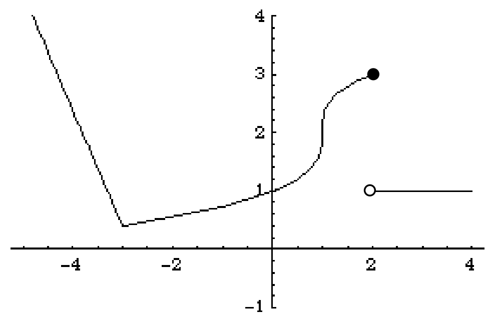
(a) $(g\circ f)(x) = g(f(x)) = g(3x-5) = (3x-5)^3$, so $(g\circ f)(1) = (3(1)-5)^3 = (-2)^3 = -8$.
(b) $(g-f)(x) = g(x)-f(x) = x^3-(3x-5) = x^3-3x+5$, so $(g-f)(0) = 0^3-3(0)+5 = 5$.
(c) $(f \circ f)(x) = f(f(x)) = f(3x - 5) = 3(3x - 5) - 5 = 9x - 15 - 5 = 9x - 20$, so $(f \circ f)(2) = 9(2) - 20 = -2$.
(d) $(f \circ g)(x) = f(g(x)) = f(x^3) = 3x^3 - 5$, so $(f \circ g)(-1) = 3(-1)^3 - 5 = -3 - 5 = -8$.

2. $\lim_{x \to 5} \dfrac{x-5}{x^2-12x+35} =$ $\underline{\quad -\dfrac{1}{2} \quad}$

We have

$$\begin{aligned}\lim_{x \to 5} \frac{x-5}{x^2-12x+35} &= \lim_{x \to 5} \frac{x-5}{(x-5)(x-7)} \\ &= \lim_{x \to 5} \frac{1}{x-7} = -\frac{1}{2}.\end{aligned}$$

3. Use the graph of $g(t)$ shown at right to answer parts (3a) and (3b). For each question, list **all** the t-values or largest intervals that make the sentence true.



(a) The value(s) of t at which $g(t)$ is not continuous is/are $\underline{\quad 2 \quad}$.

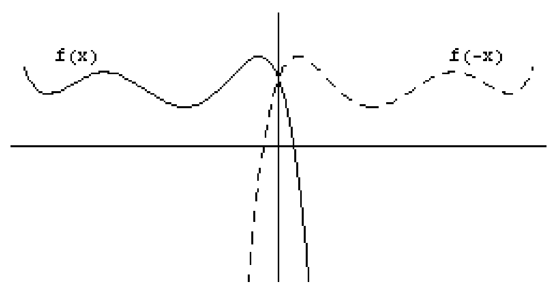
The only place I am required to "pick up my pencil" is at $t = 2$.

(b) The interval(s) on which $g(t)$ is continuous is/are $\underline{\quad (-\infty, 2] \text{ and } (2, \infty) \quad}$.

$g(t)$ is continuous everywhere except at $t = 2$. At $t = 2$, $g(t)$ is continuous from the left.

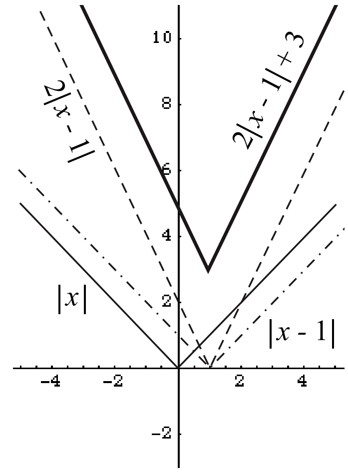
Graphs.

1. The graph of $f(x)$ is shown at right. On the same axes, sketch the graph of $f(-x)$.



The transformation $f(x) \to f(-x)$ results in a horizontal reflection of the graph about the y-axis.

2. On the axes at right, sketch the graph of $h(x) = 2|x-1|+3$.

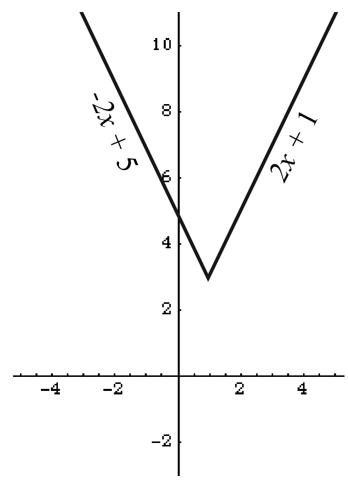


There are two ways to do this problem. You may consider transformations of the graph of $|x|$ as follows:

$$|x| \xrightarrow[\text{right 1}]{\text{shift}} |x-1| \xrightarrow[\text{vertically by 2}]{\text{stretch}} 2|x-1| \xrightarrow[\text{up 3}]{\text{shift}} 2|x-1|+3$$

(there are other orders possible for the sequence of transformations).

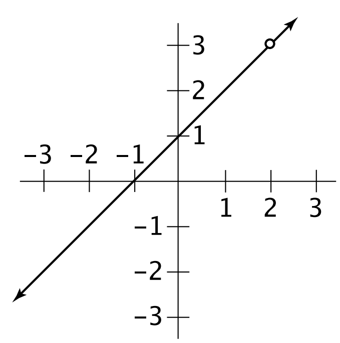
Or, you can write the function as a piecewise function and then graph the two lines:



$$h(x) = \begin{cases} 2(x-1)+3 & \text{if } x-1 \geq 0 \\ \\ -2(x-1)+3 & \text{if } x-1 < 0 \end{cases}$$
$$= \begin{cases} 2x+1 & \text{if } x \geq 1 \\ -2x+5 & \text{if } x < 1 \end{cases}$$

(Reality check: notice that the slopes of the two lines are ± 2, and that we change rules at $x = 1$, as we expected from the first solution.) $h(1) = 2(1)+1 = 3$, so the vertex is at $(1,3)$, and we get the graph shown at right.

3. On the axes at right, sketch a graph of $g(t) = \dfrac{t^2 - t - 2}{t-2}$.



Note that

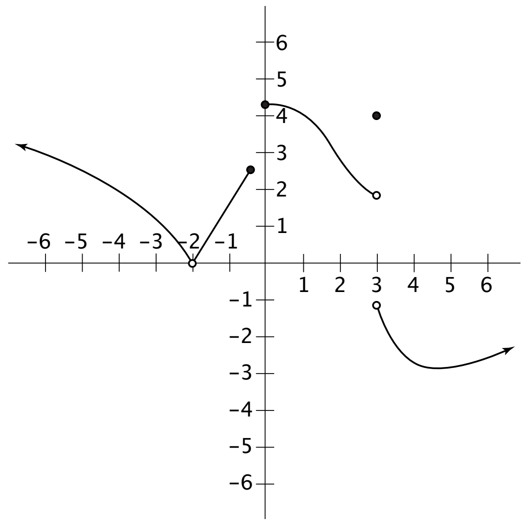
$$g(t) = \frac{(t-2)(t+1)}{t-2} = t+1$$

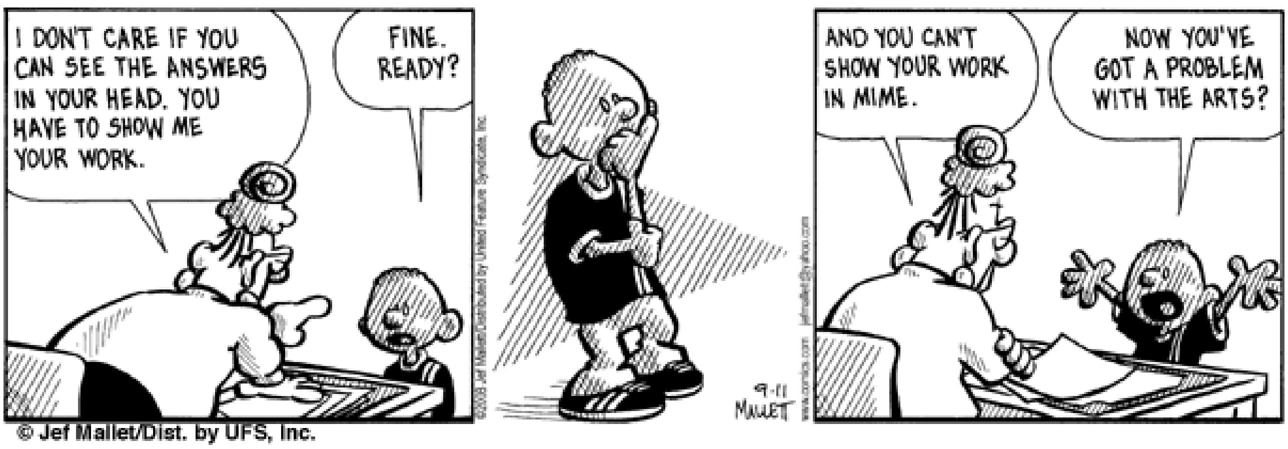
(as long as $t \neq 2$). Therefore the graph of $g(t)$ is identical to the graph of the line $y = t+1$, except that there is a hole at $t = 2$.

4. On the axes at right, sketch a graph of any function $f(x)$ satisfying all of the following:

   - $\lim_{x\to 3^-} f(x) = 2$
   - $\lim_{x\to 3^+} f(x) = -1$
   - $f(3) = 4$
   - $\lim_{x\to -2} f(x) = 0$
   - $\lim_{x\to 0} f(x)$ does not exist

   There are many possible solutions. Here is one.





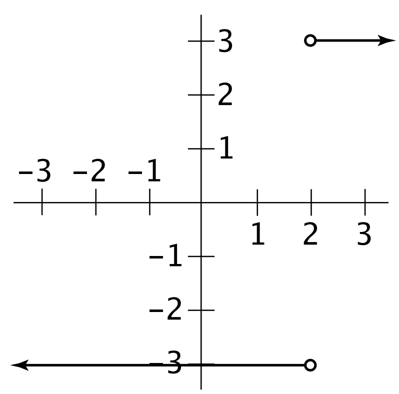
Work and Answer. *You must show all relevant work to receive full credit.*

1. Write $f(x) = \dfrac{|3x-6|}{x-2}$ as a piecewise function and graph the function. What is the domain of $f(x)$?

$$f(x) = \begin{cases} \dfrac{3x-6}{x-2} & \text{if } 3x-6>0 \\ \\ \dfrac{-(3x-6)}{x-2} & \text{if } 3x-6<0 \end{cases}$$

$$= \begin{cases} \dfrac{3(x-2)}{x-2} & \text{if } 3x>6 \\ \\ \dfrac{-3(x-2))}{x-2} & \text{if } 3x<6 \end{cases}$$

$$= \begin{cases} 3 & \text{if } x>2 \\ -3 & \text{if } x<2 \end{cases}$$



The graph has a break at $x = 2$. It looks like the horizontal line $y = 3$ for $x > 2$, but looks like $y = -3$ for $x < 2$. The function is undefined at $x = 2$. In fact the domain is $\boxed{\text{all real numbers except } 2}$

2. Find the domain of the function $g(x) = \dfrac{3\sqrt{x}}{4x-1}$. *Express your answer in interval notation.*

   Because of the denominator, we know $4x - 1 \neq 0$. Solving for x, we get $4x \neq 1$, or $x \neq \dfrac{1}{4}$. We also have, because of the square root, $x \geq 0$. So the domain is all real numbers greater than or equal to 0, except for $\frac{1}{4}$. In interval notation, this is $\boxed{\left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)}$

3. Compute $\displaystyle\lim_{x\to 0^-} \frac{|x| - x}{x}$. If the limit does not exist, explain why.

   When $x < 0$, $|x| = -x$. In this case $\dfrac{|x| - x}{x} = \dfrac{-x - x}{x} = \dfrac{-2x}{x} = -2$. This tells us that the graph of $\dfrac{|x| - x}{x}$ to the left of the y-axis looks like the horizontal line $y = -2$. Since in this limit x approaches 0 only from the left (i.e. from the negative side), the answer is

$$\lim_{x\to 0^-} \frac{|x| - x}{x} = \boxed{-2}$$

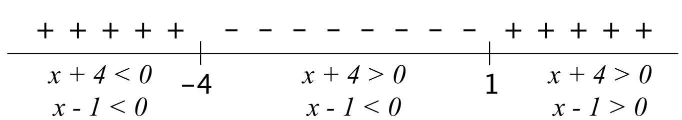
4. Find the domain of the function $f(x) = \sqrt[20]{-x^2 - 3x + 4}$. *Express your answer in interval notation.*

   We know $-x^2 - 3x + 4 \geq 0$ since it is under an even root. Solving the inequality we have

$$x^2 + 3x - 4 \leq 0 \qquad \text{(we divided both sides by } -1 \text{ to make factoring easier; remember that this switches the direction of the inequality)}$$

$$(x + 4)(x - 1) \leq 0$$

   We know that $x = -4$ and $x = 1$ are the two numbers that make the left side *equal* to 0. Now we need to find all the numbers that make the left side *less than* 0. The easiest way to do this is to draw a number line with -4 and 1 on it, and test points in between and on either side.



   You can see that in the three areas we tested, only the middle (between -4 and 1) comes out negative, as desired. So the set of all solutions to the inequality $x^2 + 3x - 4 \leq 0$, and therefore the domain of $f(x)$, is $-4 \leq x \leq 1$, or in interval notation, $\boxed{[-4, 1]}$