Math 76 Practice Problems for Midterm II – Solutions
§§6.4-8.1

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. $\displaystyle\int_0^e \ln x\, dx =$

(a) 1

$\boxed{\text{(b)}}$ 0

(c) ∞

(d) $-\infty$

This is an improper integral. We have

$$\begin{aligned}
\int_0^e \ln x\, dx &= \lim_{t\to 0^+} \int_t^e \ln x\, dx \\
&= \lim_{t\to 0^+} x\ln x - x \Big|_t^e \qquad \text{(using integration by parts)} \\
&= \lim_{t\to 0^+} (e-e) - (t\ln t - t) \\
&= \lim_{t\to 0^+} -t\ln t + t \\
&= \lim_{t\to 0^+} -t\ln t
\end{aligned}$$

Recall from Math 75 that this is an indeterminate form of type $0\cdot -\infty$. So we use l'Hôpital's Rule as follows:

$$\begin{aligned}
&= \lim_{t\to 0^+} -\frac{\ln t}{\frac{1}{t}} \\
&= \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{1}{t^2}} \\
&= \lim_{t\to 0^+} t = \boxed{0}
\end{aligned}$$

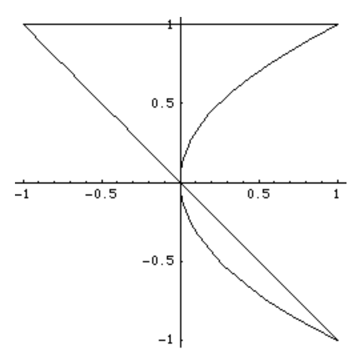
2. What expression best represents the area between $x = y^2$ and $x = -y$ from $y = -1$ to $y = 1$?

(a) $\displaystyle\int_{-1}^0 (y^2+y)\, dy + \int_0^1 (-y-y^2)\, dy$

(b) $\boxed{\displaystyle\int_{-1}^0 (-y-y^2)\, dy + \int_0^1 (y^2+y)\, dy}$

(c) $\displaystyle\int_{-1}^1 (y^2+y)\, dy$

(d) $\displaystyle\int_{-1}^0 (y^2-y)\, dy + \int_0^1 (y-y^2)\, dy$



The region described is in two pieces, as shown. The two curves cross at $y = 0$.

From $y = -1$ to $y = 0$, $x = -y$ is on the right. From $y = 0$ to $y = 1$, $x = y^2$ is on the right.

Therefore the area is

$$\int_{-1}^{0} (-y - y^2)\, dy \; + \int_{0}^{1} (y^2 - (-y))\, dy$$
$$= \int_{-1}^{0} (-y - y^2)\, dy \; + \int_{0}^{1} (y^2 + y)\, dy.$$

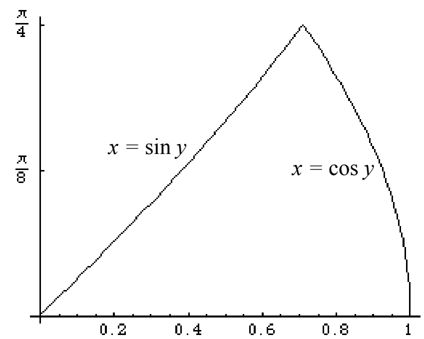
3. The volume of the solid formed by rotating the region shown about the y-axis is

(a) $2\pi \displaystyle\int_0^{\pi/4} y\,(\sin y - \cos y)\; dy$

(b) $\pi \displaystyle\int_0^{\pi/4} (\cos y - \sin y)^2\; dy$

(c) $\boxed{\pi \displaystyle\int_0^{\pi/4} \left(\cos^2 y - \sin^2 y\right)\; dy}$

(d) $2\pi \displaystyle\int_0^{\pi/4} y\,(\cos y - \sin y)\; dy$



Since the region is formed by functions of y and is being rotated about a vertical axis, we use the **disk method**:

At any y between 0 and $\dfrac{\pi}{4}$, the outer radius of the disk is $R = \cos y$ and the inner radius of the disk is $r = \sin y$. Therefore the volume is

$$\pi \int_0^{\pi/4} \left((\cos y)^2 - (\sin y)^2\right)\; dy$$
$$= \pi \int_0^{\pi/4} \left(\cos^2 y - \sin^2 y\right)\; dy.$$

4. The volume of the solid formed by rotating the region enclosed by the curves $y = \dfrac{1}{x^3}$, $y = \dfrac{1}{x^2}$, and $x = 2$ about the line $x = -1$ is

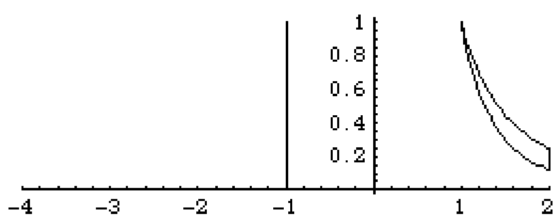
(a) $2\pi \displaystyle\int_0^2 (x+1)\left(\frac{1}{x^3} - \frac{1}{x^2}\right) dx$

(b) $\boxed{2\pi \displaystyle\int_1^2 (x+1)\left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx}$

(c) $2\pi \displaystyle\int_0^2 (x-1)\left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

(d) $2\pi \displaystyle\int_1^2 (1-x)\left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

The region being rotated is shown at right with the axis of rotation. It is the same region as in Work and Answer #1. Since the region is formed from functions of x and is being rotated about a vertical axis, we use the **shell method**:



At any x between 1 and 2, the height of the shell is $h = \dfrac{1}{x^2} - \dfrac{1}{x^3}$ and the radius is $r = x + 1$. Therefore the volume is

$$2\pi \int_1^2 (x+1)\left(\frac{1}{x^2} - \frac{1}{x^3}\right)\, dx.$$

5. Lois Lane, whose mass is 50 kg, is hanging from a 20-meter rope tied to a crane. Superman is at the top of the crane. In order to rescue Lois, he must pull the rope all the way up to the top of the crane. If the rope has a mass of 10 kg, then the work Superman must do in order to rescue Lois is

   (a) 10,780 N

   (b) $\boxed{\text{10,780 J}}$

   (c) 9,800 N

   (d) 9,800 J

The sneaky way to determine the answer is to notice that

- The work done (metric system) is measured in Joules (J), so the answer is either (b) or (d).
- Lois Lane's weight is $50 \cdot 9.8 = 490$ N, so the work required to lift only her is $490 \cdot 20 = 9800$ J, since the rope is 20 m long. So the answer must be (b) since Superman also has to pull the rope up!

But here's how to do the integral:

The rope weighs $10 \cdot 9.8 = 98$ N, or $\dfrac{98}{20} = 4.9$ Newtons per meter. So if Superman has pulled up x meters of rope, the weight of the rope he has pulled up is $4.9x$. Therefore the weight he is still pulling is $98 - 4.9x = 4.9(20 - x)$ Newtons, in addition to Lois's 490 N. The total work done, then, is

$$\begin{aligned}
W &= \int_0^{20} (4.9(20-x) + 490)\ dx \\
&= 4.9 \int_0^{20} (20 - x + 100)\ dx \\
&= 4.9 \int_0^{20} (120 - x)\ dx \\
&= 4.9 \left(120x - \frac{1}{2}x^2\right)\Bigg|_0^{20} \\
&= 4.9\left(120 \cdot 20 - \frac{1 \cdot 20^2}{2}\right) - (0 - 0) \\
&= 4.9(2400 - 200) \\
&= 4.9(2200) \\
&= \boxed{10,780 \text{ J}}
\end{aligned}$$

6. A rectangular aquarium 4 ft. wide, 6 ft. long, and 2 ft. high is full of water. If a pump is placed at the top of the tank, the work done in pumping *half* the water out is

   (a) 62.5(6) ft.-lb.

   (b) $\boxed{\text{62.5(12) ft.-lb.}}$

   (c) 62.5(24) ft.-lb.

   (d) 62.5(48) ft.-lb.

Using the formula $W = \omega \displaystyle\int_0^b (x + P)A(x)\ dx$ and the weight of water $\omega = 62.5$ lb./ft.^{3}, we get the integral

$$W = 62.5 \int_0^1 (x + 0)24\ dx,$$

since we are pumping water from a depth of 0 ft. to a depth of 1 ft. (half the water in the tank). $P = 0$ since the pump is at the top of the tank, and $A(x) = 6 \cdot 4 = 24$ at all depths x. Evaluating the above integral, we get

$$\begin{aligned} 62.5 \int_0^1 24x \, dx &= 62.5 \cdot 12x^2 \Big|_0^1 \\ &= 62.5 \cdot 12(1^2 - 0^2) = \boxed{62.5(12) \text{ ft.-lb.}} \end{aligned}$$

7. The length of the curve $x = y^3 - y$ from $y = 1$ to $y = 3$ is

(a) $2\pi \int_1^3 \sqrt{1 + (3y^2 - 1)^2} \, dy$

$\boxed{\text{(b)}}$ $\int_1^3 \sqrt{9y^4 - 6y^2 + 2} \, dy$

(c) $\int_1^3 \sqrt{1 + y^3 + y} \, dy$

(d) $\int_1^3 \sqrt{3y^2} \, dy$

We have $x' = 3y^2 - 1$, so $(x')^2 = 9y^4 - 6y^2 + 1$. Therefore the arc length is as given above.

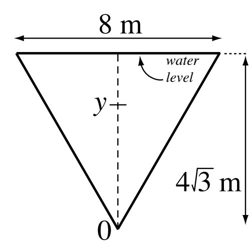
8. A trough is filled with water. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. The hydrostatic force on one end of the trough is

(a) $\dfrac{9800\sqrt{3}}{2} \int_0^{4\sqrt{3}} y(y-8) \, dy$

(b) $\dfrac{9800}{\sqrt{3}} \int_0^{4\sqrt{3}} y^2 \, dy$

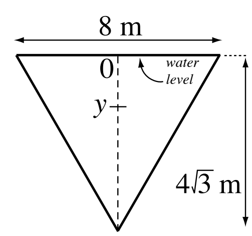
(c) $9800 \int_0^8 (8 - y)y \, dy$

$\boxed{\text{(d)}}$ $\dfrac{19600}{\sqrt{3}} \int_0^{4\sqrt{3}} (4\sqrt{3} - y)y \, dy$



The triangle is shown. Using the Pythagorean Theorem the height of the triangle is $4\sqrt{3}$ m, so putting the origin at the bottom vertex of the triangle we have that the depth at y is $d(y) = 4\sqrt{3} - y$. Using similar triangles we also find that $w(y) = \dfrac{2y}{\sqrt{(3)}}$. So we have

$$F = 9800 \int_0^{4\sqrt{3}} (4\sqrt{3} - y) \frac{2y}{\sqrt{(3)}} \, dy = \frac{19600}{\sqrt{3}} \int_0^{4\sqrt{3}} (4\sqrt{3} - y)y \, dy.$$



One note: if you put the origin somewhere else your integral will look different, but you should still be able to wiggle it (through algebra, substitution, or other means) into something that matches the above. For instance, if we put the origin at the top of the triangle (as at right), then $d(y) = y$ and $w(y) = \dfrac{2}{\sqrt{3}}(4\sqrt{3} - y)$, which simplifies to $w(y) = 8 - \dfrac{2}{\sqrt{3}}y$. The integral still goes from 0 to $4\sqrt{3}$, as it turns out.

Therefore we get $F = 9800 \int_0^{4\sqrt{3}} \left(8 - \dfrac{2}{\sqrt{3}}y\right) y \, dy = 9800 \int_0^{4\sqrt{3}} \dfrac{2}{\sqrt{3}} \left(4\sqrt{3} - y\right) y \, dy$, which is now the same as the answer in the multiple choice.

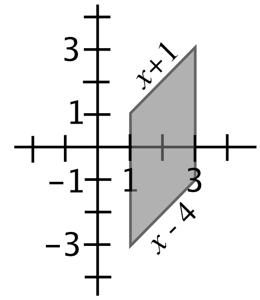
9. The coordinates of the center of mass of the region enclosed by $y = x$, $y = x - 4$, $x = 1$ and $x = 3$ are

(a) $(2.5, 0.5)$

(b) $(1.75, 0)$

$\boxed{\text{(c)}}$ $(2, 0)$

(d) $(2, 0.5)$



A picture of the region is shown. It is a parallelogram. By inspection, we can see by a kind of "skew symmetry" that the coordinates must be as above. But we can also compute them using the formulas: the area of a parallelogram is the base times the height, so $4 \cdot 2 = 8$ (or you can do the integral $\int_1^3 (x - (x-4))\,dx$). So the coordinates are

$$\begin{aligned}
\overline{x} &= \frac{1}{8}\int_1^3 x(x-(x-4))\,dx \\
&= \frac{1}{8}\int_1^3 x(4)\,dx = \frac{1}{8}\int_1^3 4x\,dx \\
&= \frac{1}{8}2x^2\Big|_1^3 = \frac{1}{8}(18-2) = \frac{16}{8} = 2
\end{aligned}$$

and

$$\begin{aligned}
\overline{y} &= \frac{1}{8}\cdot\frac{1}{2}\int_1^3 (x^2 - (x-4)^2)\,dx \\
&= \frac{1}{16}\int_1^3 x^2 - (x^2 - 8x + 16)\,dx \\
&= \frac{1}{16}\int_1^3 8x - 16\,dx \\
&= \frac{1}{16}(4x^2 - 16x)\Big|_1^3 \\
&= \frac{1}{16}((36-48) - (4-16)) = 0.
\end{aligned}$$

10. The nth term of the sequence $\{-3, 4, 11, 18, 25, \ldots\}$, counting $a_1 = -3$ as the first term, is

(a) $a_n = 5n - 2$

$\boxed{\text{(b)}}$ $a_n = 7n - 10$

(c) $a_n = n^2 - 4$

(d) $a_n = -3n + 7$

This is an arithmetic sequence with a common difference of 7, so only (b) can be correct. Sure enough, when $n = 1$, $a_1 = 7n - 10 = 7 - 10 = -3$.

Fill-In.

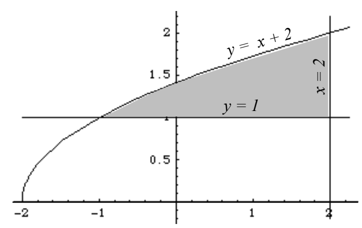
1. $\displaystyle\int_1^\infty \frac{5}{x^3}\,dx = \underline{\quad \frac{5}{2} \quad}$.

   We have

$$\begin{aligned}\int_1^\infty \frac{5}{x^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{5}{x^3}\,dx \\ &= \lim_{t\to\infty} -\frac{5}{2x^2}\bigg|_1^t \\ &= \lim_{t\to\infty} -\frac{5}{2t^2}+\frac{5}{2} = \frac{5}{2}.\end{aligned}$$

2. If the region enclosed by the curves $y=\sqrt{x+2}$, $y=1$ and $x=2$ is rotated about the x-axis, the volume of the resulting solid is $\underline{\frac{9\pi}{2}}$.

   The region is shown at right. Since we are rotating a region formed from functions of x about a horizontal axis, it is easiest to use **disks**. The region goes from $x=-1$ to $x=2$. Also note that $R=\sqrt{x+2}$ and $r=1$. Thus we get



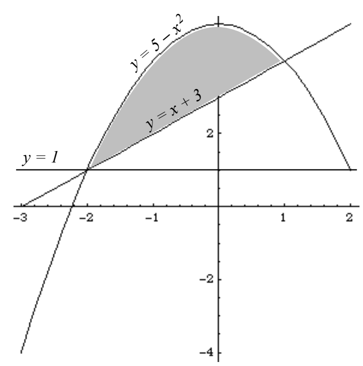
$$\begin{aligned}V &= \pi\int_{-1}^2 (\sqrt{x+2})^2 - 1^2\,dx \\ &= \pi\int_{-1}^2 (x+2-1)\,dx \\ &= \pi\int_{-1}^2 (x+1)\,dx\end{aligned}$$

$$\begin{aligned}&= \pi\left(\frac{1}{2}x^2+x\right)\bigg|_{-1}^2 \\ &= \pi\left(2+2-\left(\frac{1}{2}-1\right)\right) \\ &= \pi\left(\frac{9}{2}\right) = \boxed{\frac{9\pi}{2}}\end{aligned}$$

   This region can also be rewritten in terms of y and the problem solved using shells. See me for help if you want to go over this.

3. If the region enclosed by the curves $y=5-x^2$ and $y=x+3$ is rotated about the line $y=1$, the volume of the resulting solid is $\underline{\frac{108\pi}{5}}$.

   The region is shown at right. Since we are rotating a region formed from functions of x about a horizontal axis, it is easiest to use **disks**. To find where the curves intersect, we set them equal to each other and solve for x:



$$\begin{aligned}5-x^2 &= x+3 \\ x^2+x-2 &= 0 \\ (x+2)(x-1) &= 0 \\ x=-2 \quad &, \quad x=1\end{aligned}$$

Therefore the region goes from $x = -2$ to $x = 1$. Also note that $R = (5 - x^2) - 1 = 4 - x^2$ and $r = (x + 3) - 1 = x + 2$. Thus we get

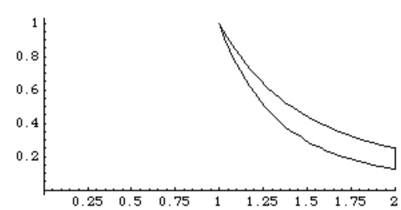
$$\begin{aligned}
V &= \pi \int_{-2}^{1} (4 - x^2)^2 - (x+2)^2 \, dx \\
&= \pi \int_{-2}^{1} (16 - 8x^2 + x^4 - (x^2 + 4x + 4)) \, dx \\
&= \pi \int_{-2}^{1} (12 - 4x - 9x^2 + x^4) \, dx = \pi \left(12x - 2x^2 - 3x^3 + \frac{1}{5}x^5 \right) \Bigg|_{-2}^{1} \\
&= \pi \left(\left(12 - 2 - 3 + \frac{1}{5} \right) - \left(-24 - 8 + 24 - \frac{32}{5} \right) \right) \\
&= \pi \left(7 + \frac{1}{5} + 8 + \frac{32}{5} \right) = \pi \left(15 + \frac{33}{5} \right) = \boxed{\frac{108\pi}{5}}
\end{aligned}$$

4. If 25 N of force are required to keep a spring stretched 20 cm beyond its natural length, then the spring constant for the spring is $k =$ 125 N/m .

   We use Hooke's Law $F(x) = kx$. First we must convert 20 cm to $0.2 = \frac{1}{5}$ m. Then $25 = k \cdot \frac{1}{5}$. Therefore $k = 25 \cdot 5 = \boxed{125 \text{ N/m}}$

Work and Answer. *You must show all relevant work to receive full credit.*

1. Find the area enclosed by the curves $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, and $x = 2$.

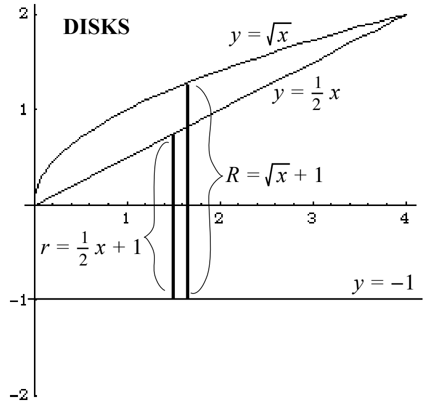
   

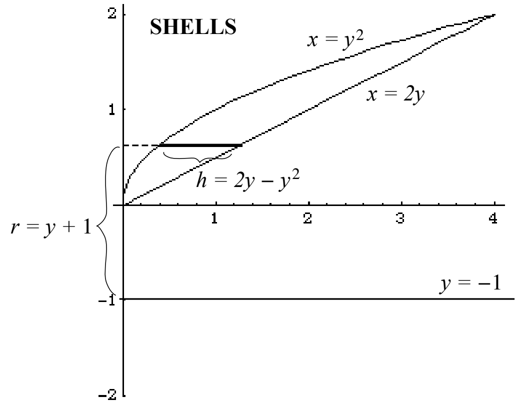
   The region described is shown. It is the same region as in Multiple Choice #4.
   Notice that the curve $\frac{1}{x^2}$ is on top between $x = 1$ and $x = 2$. Therefore the area is

$$\begin{aligned}
\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3} \right) dx &= \int_1^2 (x^{-2} - x^{-3}) \, dx \\
&= -x^{-1} + \frac{1}{2}x^{-2} \Bigg|_1^2 = -\frac{1}{x} + \frac{1}{2x^2} \Bigg|_1^2 \\
&= \left(-\frac{1}{2} + \frac{1}{8} \right) - \left(-1 + \frac{1}{2} \right) = \boxed{\frac{1}{8}}
\end{aligned}$$

2. (a) Use the disk method to find the volume of the solid formed by rotating the region enclosed by the curves $x = y^2$ and $x = 2y$ about the line $y = -1$.

   Since we are rotating about a horizontal axis, we will need to rewrite the curves in terms of x. The curves intersect at the points $(0, 0)$ and $(4, 2)$, so we will only need the upper half of the parabola $x = y^2$. Therefore we can rewrite this as $y = \sqrt{x}$ (the positive square root).





The region is shown above left, along with R and r. Thus the volume is

$$\begin{aligned}
V &= \pi \int_0^4 \left(\sqrt{x}+1\right)^2 - \left(\frac{1}{2}x+1\right)^2 \, dx \\
&= \pi \int_0^4 \left(x+2\sqrt{x}+1\right) - \left(\frac{1}{4}x^2+x+1\right) \, dx \\
&= \pi \int_0^4 \left(-\frac{1}{4}x^2+2\sqrt{x}\right) \, dx = \pi \left(-\frac{1}{12}x^3+\frac{4}{3}x^{3/2}\right) \Bigg|_0^4 \\
&= \pi \left(\left(-\frac{1}{12}\cdot 4^3 + \frac{4}{3}\cdot 4^{3/2}\right) - (0+0) \right) \\
&= \pi \left(-\frac{16}{3}+\frac{32}{3}\right) = \boxed{\frac{16\pi}{3}}
\end{aligned}$$

(b) Use the shell method to find the volume of the solid formed by rotating the region enclosed by the curves $x = y^2$ and $x = 2y$ about the line $y = -1$.

Since we are rotating about a horizontal axis, we can leave the curves in terms of y. The curves intersect at the points $(0,0)$ and $(4,2)$, as before. The region is shown above right, along with r and h. Thus the volume is

$$\begin{aligned}
V &= 2\pi \int_0^2 (y+1)(2y-y^2) \, dy \\
&= 2\pi \int_0^2 (2y^2+2y-y^3-y^2) \, dy \\
&= 2\pi \int_0^2 (-y^3+y^2+2y) \, dy \\
&= 2\pi \left(-\frac{1}{4}y^4+\frac{1}{3}y^3+y^2\right) \Bigg|_0^2 \\
&= 2\pi \left(\left(-\frac{1}{4}\cdot 2^4 + \frac{1}{3}\cdot 8 + 2^2\right) - (0+0+0) \right) \\
&= 2\pi \left(-4+\frac{8}{3}+4\right) = \boxed{\frac{16\pi}{3}}
\end{aligned}$$

(c) Should the answers to (a) and (b) be the same? Why or why not?

In (a) and (b) we are rotating the same region about the same axis, so the resulting solids should be the same. Therefore the volumes should be equal.

3. A certain spring has a natural length of 18 in. If 10 lb. of force is needed to keep the spring stretched to a length of 24 in., what is the work done in stretching it to 36 in.?

   This is a problem where the units are in the English system. However, the distance units are in inches, not feet. So the first thing to do is convert the distances to feet: we have

$$\begin{aligned} 18\,\text{in.} &= \tfrac{3}{2}\,\text{ft.} \\ 24\,\text{in.} &= 2\,\text{ft.} \\ 36\,\text{in.} &= 3\,\text{ft.} \end{aligned}$$

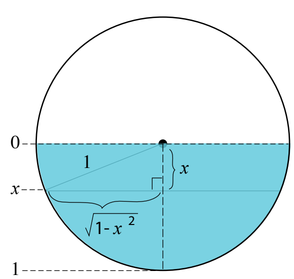
   Next we use Hooke's Law $F(x) = kx$. We need to use the information in the problem to find k. The problem says that 10 lb. of force are needed to stretch the spring $2 - \dfrac{3}{2} = \dfrac{1}{2}$ ft. (remember that x in Hooke's Law is the number of feet ***beyond** the natural length*). So $10 = k \cdot \dfrac{1}{2}$. Therefore $k = 20$. So the work done to stretch it to 3 ft. (= 1.5 ft. beyond the natural length) is

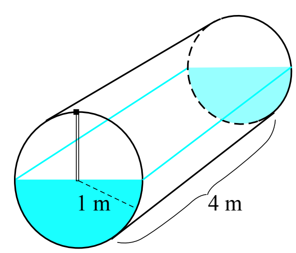
$$\begin{aligned} W &= \int_0^{1.5} 20x\,dx \\ &= 10x^2 \Big|_0^{1.5} \\ &= 10 \cdot (1.5)^2 - 0 = 10 \cdot 2.25 = \boxed{22.5\,\text{ft.-lb.}} \end{aligned}$$

4. A tank in the shape of a cylinder on its side is half full of water. A pump is at the top of the tank, as shown below.

   (a) Find the hydrostatic force on one of the circular sides of the tank.

   It is easiest to set up our coordinate system so that the line $x = 0$ passes through the center of the circle, as shown.





   The depth at x is $d(x) = x$. We also have $w(x) = 2\sqrt{1 - x^2}$ (using the Pythagorean Theorem; see the picture). Therefore the hydrostatic force is

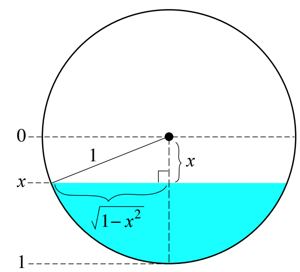
$$\begin{aligned} H.F. &= 9800 \int_0^1 x \cdot 2\sqrt{1 - x^2}\,dx \\ &= 9800 \cdot 2 \int_0^1 x\sqrt{1 - x^2}\,dx \end{aligned}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$, and we get

$$\begin{aligned}
&= -9800 \cdot 2 \cdot \frac{1}{2} \int_?^? \sqrt{u}\,du \\
&= -9800 \cdot \frac{2}{3} u^{3/2} \Big|_?^? \\
&= -\frac{19600}{3} (1 - x^2)^{3/2} \Big|_0^1 \\
&= -\frac{19600}{3} (0^{3/2} - 1^{3/2}) = \boxed{\frac{19600}{3} \text{ N}}
\end{aligned}$$

(b) Set up, **but do not evaluate,** an integral for the work done in pumping all the water out of the tank.

We set $x = 0$ to be the initial water level, as shown above. The pump is 1 m above that, so $P = 1$ (see the formula $W = \omega \int_0^b (x + P)A(x)\,dx$ on the formula list). Since we are in the metric system, $\omega = 9800$. Finally, the surface area $A(x)$ of the water at each depth x is a rectangle 4 m long and $2\sqrt{1-x^2}$ m wide (similar to part (a); see the picture), so $A(x) = 8\sqrt{1-x^2}$ square meters. Therefore the work done is



$$\begin{aligned}
W &= 9800 \int_0^1 (x+1) \cdot 8\sqrt{1-x^2}\,dx \\
&= \boxed{78,400 \int_0^1 (x+1)\sqrt{1-x^2}\,dx \text{ J}}
\end{aligned}$$

In case you are interested in evaluating the above integral (it would be great practice!), here's the solution:
First distribute the $x + 1$ to get

$$78,400 \left[\int_0^1 x\sqrt{1-x^2}\,dx + \int_0^1 \sqrt{1-x^2}\,dx \right].$$

For the first integral we use a u-substitution: let $u = 1 - x^2$. Then $du = -2x\,dx$. We also need to change the x-limits to u-limits. When $x = 0$, $u = 1 - 0^2 = 1$, and when $x = 1$, $u = 1 - 1^2 = 0$. Therefore we get

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{2} \int_1^0 \sqrt{u}\,du \\
&= \frac{1}{2} \int_0^1 \sqrt{u}\,du \qquad \text{(switching the limits and getting rid of the negative)} \\
&= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_0^1 \\
&= \frac{1}{3}(1 - 0) = \frac{1}{3}.
\end{aligned}$$

For the second integral we can use geometry. $\int_0^1 \sqrt{1-x^2}\,dx$ represents the area of $\frac{1}{4}$ of a circle of radius 1, so

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\cdot\pi\cdot 1^2 = \frac{\pi}{4}.$$

Therefore the final answer for the work done would be

$$W = 78,400\left(\frac{1}{3}+\frac{\pi}{4}\right)\ \text{J}.$$

5. Find the length of the curve $f(x) = \dfrac{e^x+e^{-x}}{2}$ from $x=0$ to $x=1$.

We have $f'(x) = \dfrac{e^x-e^{-x}}{2}$, so $1+(f'(x))^2 = 1+\frac{1}{4}(e^{2x}-2+e^{-2x}) = \frac{1}{4}(e^{2x}+2+e^{-2x})$ (I'm skipping a couple of algebra steps here), which equals $\left(\frac{1}{2}(e^x+e^{-x})\right)^2$. This is one of those fancy integrals where the middle term under the square root changes sign and ends up forming a perfect square again after the "1+" part is added in. Therefore we have

$$\begin{aligned}
L &= \int_0^1 \sqrt{\left(\frac{1}{2}(e^x+e^{-x})\right)^2}\,dx \\
&= \frac{1}{2}\int_0^1 (e^x+e^{-x})\,dx \\
&= \frac{1}{2}(e^x-e^{-x})\Big|_0^1 \\
&= \frac{1}{2}\left(\left(e-\frac{1}{e}\right)-(1-1)\right) = \boxed{\frac{1}{2}\left(e-\frac{1}{e}\right)}
\end{aligned}$$

6. Find the length of the curve $x = \dfrac{y^2-1}{2}$ from $y=1$ to $y=3$.

We have $x' = \frac{1}{2}\cdot 2y = y$; therefore $1+(x')^2 = 1+y^2$, and the length of the curve is

$$L = \int_1^3 \sqrt{1+y^2}\,dy$$

Using integral #21 in the tables (this would be provided to you in the exam), or trigonometric substitution, we get

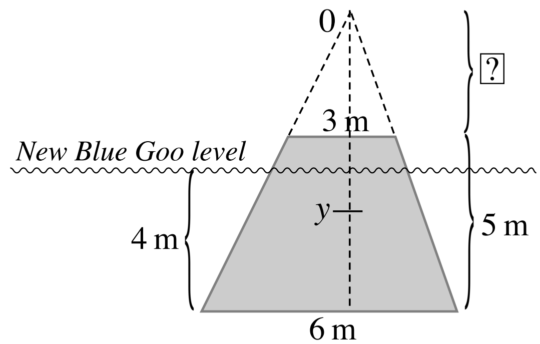
$$\begin{aligned}
&= \frac{y}{2}\sqrt{1+y^2}+\frac{1}{2}\ln(y=\sqrt{1+y^2})\Big|_1^3 \\
&= \frac{3}{2}\sqrt{10}+\frac{1}{2}\ln(3+\sqrt{10}) - \left(\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln(1+\sqrt{2})\right) \\
&= \boxed{\frac{1}{2}\left(3\sqrt{10}-\sqrt{2}+\ln\left(\frac{3+\sqrt{10}}{1+\sqrt{2}}\right)\right)}
\end{aligned}$$

Remark. You may leave your answer as in the second to last line above, if you wish. As always, the rule for this class is *simplify at your own risk.*

7. Find the hydrostatic force on the wall shown. The fluid is New Blue Goo (density 1500 kg/m^3).

There are several ways to do this problem. But, there is really no principle of "similar trapezoids" like there is with similar triangles. We can, however, use similar triangles if we add a "top" to our trapezoid as shown.

I have chosen to make the top of the triangle my origin. The first thing we need to do is figure out what $\boxed{?}$ is. Using similar triangles, we have $\dfrac{\boxed{?}+5}{6} = \dfrac{\boxed{?}}{3}$, so $\boxed{?} = 5$. Now using similar triangles again we have $w(y) = \dfrac{3y}{\boxed{?}} = \dfrac{3y}{5}$ (I'm skipping some steps here). We also have $d(y) = y - 6$ (again using the top of the "triangle" as the origin).



Therefore the hydrostatic force is

$$
\begin{aligned}
F &= 1500 \cdot 9.8 \int_6^{10} \frac{3y}{5}(y-6)\,dy \\
&= \frac{1500 \cdot 9.8 \cdot 3}{5} \int_6^{10} y(y-6)\,dy \\
&= 900 \cdot 9.8 \int_6^{10} y^2 - 6y\,dy \\
&= 900 \cdot 9.8 \left(\frac{1}{3}y^3 - 3y^2\right)\Bigg|_6^{10} \\
&= 900 \cdot 9.8 \left(\left(\frac{1000}{3} - 300\right) - \left(\frac{216}{3} - 108\right)\right) \\
&= 900 \cdot 9.8 \left(\frac{1000}{3} - 300 - 72 + 108\right) = \boxed{900 \cdot 9.8 \left(\frac{1000}{3} - 264\right) \text{ N}}
\end{aligned}
$$

8. For the sequence $\{2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\}$,

(a) Find a formula for the n-th term a_n of the sequence, assuming $a_0 = 2$.

We make the following observations:

(i) The sequence is alternating, so there is a $(-1)^{\xi}$ in the formula.

(ii) The numerators are proceeding by powers of 2, so there is a $2^{\spadesuit}$ in the numerator of the formula.

(iii) The denominators are proceeding by powers of 3, so there is a $3^{\clubsuit}$ in the denominator of the formula.

We have $a_0 = 2 = +\dfrac{2^1}{3^0}$, so the formula for a_n is $\boxed{a_n = (-1)^n \dfrac{2^{n+1}}{3^n}}$

(b) *Circle the best answer.* The sequence $\{a_n\}_{n=0}^{\infty}$ above **(** $\boxed{\textbf{converges}}$ | **diverges)**.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2 \cdot \left(\dfrac{2}{3}\right)^n = 0$. So the sequence converges to 0.