Archimedean Quadrature Redux

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Archimedes' use of Eudoxos' method of exhaustion to determine the area bounded by a parabolic arc and a line segment was a crowning achievement in Greek mathematics. The promise of the method, so apparent to us now, seems to have died with Archimedes, only to rise again in different form some 1900 years later with the modern calculus. Archimedes' result though is not just about computing an area. It is about comparing a parabolic area with a related triangular area. That is, there is a geometric content in the comparison that is interesting in its own right. In recognition of this, there have been several generalizations discovered more recently that highlight the geometry using methods of modern analysis ([1], [2], [4], [6], [7] and [14]).

In this short article we would like to make the case that Archimedes' area comparisons deserve more attention, not so much because of his methods, but rather because of the interesting geometric content of the comparisons and the new questions they suggests. We feel that there are more results to be had, and present a few here with some speculation on further research directions.

History

Eudoxos of Cnidos (408-355 BC, in modern day Turkey) is generally credited with the discovery of the so-called method of exhaustion for determining the volumes of a pyramid and cone [9]. The ancient Greeks were fond of comparisons between volumes. For example, Eudoxos showed that the volume of a pyramid, respectively a cone, was one-third of the volume of the prism, respectively the cylinder, with like base and height. Although Eudoxos did not have the modern apparatus of limits, his technique amounted to approximating the volume by many simpler figures whose volumes are understood and essentially passing to a limit.

The apex of the method of exhaustion comes with Archimedes of Syracuse (287- 212 BC). Archimedes deftly used the method to prove several area and volume (circles and spheres) comparisons ([5], [8], [9], [13]). One might argue that parabolic curves are the natural next step. And a lesser mathematician of the time may have passed given the difficulty and apparent lack of obvious applications. Archimedes, however, solved the area problem for parabolas in his two related theorems (1) the quadrature of the parabola and the (2) squaring of the parabola. Both theorems compare a parabolic area to that of related triangle areas and can be found in his *Quadrature of the Parabola* and *The Method*.

The setting for Archimedes' theorems is a region in the plane bounded by a straight line segment and a parabolic arc, meeting at respective points P and Q (figure 1). In his quadrature theorem, Archimedes locates the point R on the parabolic arc that is a maximum distance, measured perpendicularly, from the line segment $\overline{PQ}$ and calls this point the *vertex* of the parabolic arc. (The tangent line to the parabolic arc at the vertex R is parallel to $\overline{PQ}$.) Quadrature states that the area bounded by the parabolic arc and the line segment is equal to $\frac{4}{3}$ of the area of $\triangle PQR$.

For his squaring of the parabola, Archimedes compares the above parabolic area to that of the so-called Archimedes triangle, $\triangle PQR'$ where R' is the intersection point of the two tangent lines to the parabola at P and Q respectively. He then goes on to prove that the parabolic area is $\frac{2}{3}$ of the area enclosed by this triangle.

Archimedes' methods for proving his theorems relied on several properties of the parabola that are not common knowledge today. (For full details of the proofs, see [5], [9] and [13].) But we do have the powerful tools of analysis that can allow us to go further.

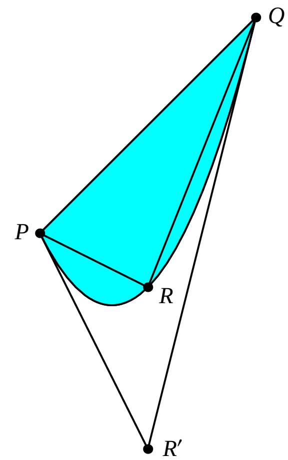

Figure 1: Archimedes' Quadrature and Squaring of the Parabola

Archimedean Quadrature and Squaring for Analytic Plane Curves

The context for our generalization will be *analytic plane curves*. A curve $\mathcal{C}$ will be called *analytic of order* n at a point $R \in \mathcal{C}$ if there is a coordinate system at R with the two respective axes tangent and normal to $\mathcal{C}$ at R so that $\mathcal{C}$ is the graph of an analytic function

$$f(x) = c_n x^n + c_{n+1} x^{n+1} + \cdots,$$

where $c_n \neq 0$. For our purposes, n will always be an even positive number. In the language of [3], the curve $\mathcal{C}$ has n-fold contact with its tangent line at R.

Note that a point R on a curve $\mathcal{C}$ is of order 2 precisely when the curvature of $\mathcal{C}$ is non-zero at R (because the curvature function is given by $\kappa(x) = f''(x)/(1+f'(x)^2)^{3/2}$). And consequently, every point on a parabolic arc is of order 2.

Let $T_R\mathcal{C}$ denote the tangent line to $\mathcal{C}$ at R. We will consider the family of triangles $\triangle PQR$ for which P and Q are on $\mathcal{C}$, where R lies between P and Q on $\mathcal{C}$, and $\overline{PQ}$ is parallel to $T_R\mathcal{C}$. This situation can be pictured as figure 1.

Let A denote the area bounded by the curve and the segment $\overline{PQ}$ (the shaded area in figure 1) and T the area of $\triangle PRQ$. Archimedes' quadrature of the parabola states that $A/T = 4/3$ if C is a parabolic arc. While we would not expect his theorem to be true for other curves, we could ask if his theorem holds "in the limit" or analytic curves. We find that the answer is yes with some additional consideration in the case of zero curvature. The following generalization was proved in the case $n = 1$ in [6].

Theorem 1 *(Generalized Archimedean Quadrature) Suppose $\mathcal{C}$ is an analytic plane curve and $R \in \mathcal{C}$ is a point of order $2n$, with $n \geq 1$, and that A and T are as described in the previous paragraph, then*

$$\lim_{\substack{PQ \to 0 \\ \overline{PQ} || T_R\mathcal{C}}} \frac{A}{T} = \frac{4n}{2n+1},$$

where the limit is taken over points $P, Q \in \mathcal{C}$ that are on opposite sides of R and $\overline{PQ}$ is parallel to the tangent line $T_R\mathcal{C}$.

Proof. By assumption, $\mathcal{C}$ is the graph of $f(x) = c_{2n}x^{2n} + c_{2n+1}x^{2n+1} + \cdots$ with $c_{2n} \neq 0$. In this coordinate system, $R = (0,0)$, $P = (a, f(a))$ and $Q = (b, f(b))$, $a < 0 < b$ and $f(a) = f(b)$. (The last point is because the line $\overline{PQ}$ us assumed to be parallel to the tangent line to $\mathcal{C}$ at R which is the horizontal axis in our coordinate system.) We may also assume $f(x) > 0$ for $x \neq 0$. By the inverse function theorem, we may write $b = \gamma(a)$ for some function γ. (See Figure 2.)

We will need to know how to compute $\gamma'(a)$ later in the proof, and this is also provided by the inverse function theorem:

$$\gamma'(a) = \frac{f'(a)}{f'(\gamma(a))}. \tag{1}$$

The area bounded by $\mathcal{C}$ and $\overline{PQ}$ is equal to

$$A = f(a)(b-a) - \int_a^b f(x)\ dx.$$

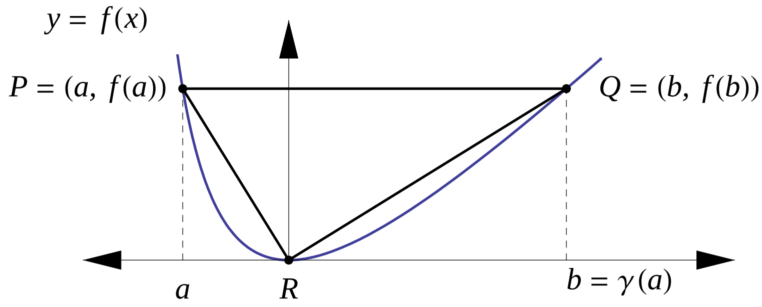

Figure 2: Generalized Archimedean Quadrature

The area enclosed by $\triangle PQR$ is

$$T = \frac{1}{2} f(a)(b-a). \tag{2}$$

Thus, the ratio A/T is

$$\begin{aligned} \frac{A}{T} &= 2\frac{f(a)(b-a) - \int_a^b f(x)\ dx}{f(a)(b-a)} \\ &= 2 - 2\left(\frac{\int_a^b f(x)\ dx}{f(a)(b-a)}\right). \end{aligned} \tag{3}$$

Letting $b = \gamma(a)$, noting that $f(\gamma(a)) = f(a)$ and using L'Hospital's rule along with the Fundamental Theorem of Calculus, we get

$$\begin{aligned} \lim_{a\to 0^-} \frac{\int_a^b f(x)\ dx}{(b-a)f(a)} &= \lim_{a\to 0^-} \frac{\int_a^{\gamma(a)} f(x)\ dx}{f(a)(\gamma(a)-a)} \\ &= \lim_{a\to 0^-} \frac{f(a)(\gamma'(a)-1)}{f'(a)(\gamma(a)-a) + f(a)(\gamma'(a)-1)} \\ &= \lim_{a\to 0^-} \frac{1}{\frac{f'(a)(\gamma(a)-a)}{f(a)(\gamma'(a)-1)} + 1}. \end{aligned} \tag{4}$$

Now focusing on the denominator term, we see that

$$\begin{aligned} \lim_{a\to 0^-} \frac{f'(a)(\gamma(a)-a)}{f(a)(\gamma'(a)-1)} &= \lim_{a\to 0^-} \left(\frac{2nc_{2n}a^{2n-1}+\cdots}{c_{2n}a^{2n}+\cdots}\right)\left(\frac{\gamma(a)-a}{\gamma'(a)-1}\right) \\ &= \lim_{a\to 0^-} \left(\frac{2nc_{2n}a^{2n}+\cdots}{c_{2n}a^{2n}+\cdots}\right)\left(\frac{\frac{\gamma(a)}{a}-1}{\gamma'(a)-1}\right). \end{aligned} \tag{5}$$

The first term in the product (5) is easily seen to approach $2n$. As for the second term, we first compute using L'Hospital's rule.

$$\lim_{a \to 0^-} \frac{\gamma(a)}{a} = \lim_{a \to 0^-} \gamma'(a).$$

And by formula (1),

$$\begin{aligned}
\lim_{a \to 0^-} \gamma'(a) &= \lim_{a \to 0^-} \frac{f'(a)}{f'(\gamma(a))} \\
&= \lim_{a \to 0^-} \frac{2nc_{2n}a^{2n-1} + \cdots}{2nc_{2n}\gamma(a)^{2n-1} + \cdots} \\
&= \lim_{a \to 0^-} \left(\frac{a}{\gamma(a)} \right)^{2n-1} \\
&= \lim_{a \to 0^-} \left(\frac{1}{\gamma'(a)} \right)^{2n-1}.
\end{aligned}$$

Upon equating the first and last term in the above string of equalities, we obtain $(\lim_{a \to 0^-} \gamma'(a))^{2n} = 1$. But $\gamma(a)$ is clearly a decreasing function of a for $a < 0$, so

$$\lim_{a \to 0^-} \frac{\gamma(a)}{a} = \lim_{a \to 0^-} \gamma'(a) = -1,$$

Substituting this into the product (5), gives us

$$\lim_{a \to 0^-} \frac{f'(a)(\gamma(a) - a)}{f(a)(\gamma'(a) - 1)} = 2n. \tag{6}$$

This can now be substituted into the fraction (4) to get

$$\lim_{a \to 0^-} \frac{\int_a^b f(x)\,dx}{(b-a)f(a)} = \frac{1}{2n+1}.$$

Finally, we have the last piece to substitute into the ratio (3), completing the proof.

$$\begin{aligned}
\lim_{a \to 0^-} \frac{A}{T} &= 2 - \frac{2}{2n+1} \\
&= \frac{4n}{2n+1}.
\end{aligned}$$

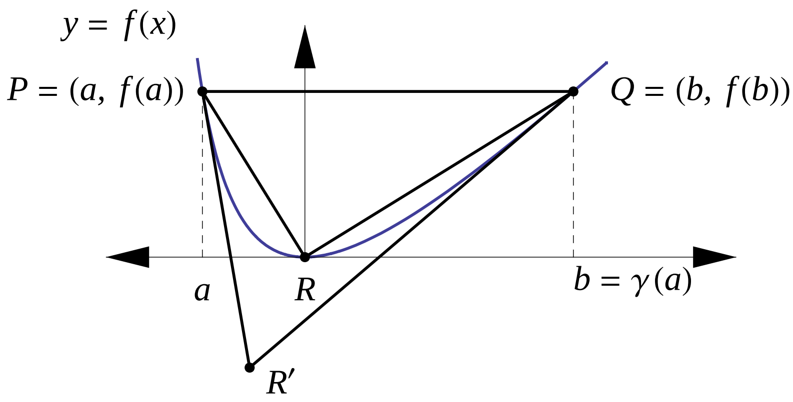

Figure 3: Generalized Archimedean Squaring

If $2n = 2$, we see that the ratio in Theorem 1 tends to $4/3$. So it would appear that there is not some other curve for which the ratios are a constant value other than $4/3$, except possibly at points of order larger than 2. In fact, this can happen. If $\mathcal{C}$ is the graph of $f(x) = x^{2n}$ and $R = (0, 0)$, then it is easy to check that A/T is constantly $4n/(2n+1)$ for $\overline{PQ}$ parallel to $T_R\mathcal{C}$. Also, the necessity of the condition $\overline{PQ}||T_R\mathcal{C}$ in the limit becomes apparent in the example $f(x) = x^2$. We leave it for the reader to verify that the limit would not exist without this extra condition.

Archimedes' squaring of the parabola can be generalized in a similar way. Using the same hypotheses as in Theorem 1, let R' be the intersection of the two tangents lines to $\mathcal{C}$ at P and Q respectively. (See Figure 3.)

Let $\tilde{T}$ be the area enclosed by $\triangle PQR'$. Using the same notation as in the previous proof, we have

$$\frac{A}{\tilde{T}} = \frac{A}{T} \cdot \frac{T}{\tilde{T}}. \tag{7}$$

It is easy to derive

$$\tilde{T} = \frac{f'(a)f'(b)(b-a)^2}{2(f'(a) - f'(b))}.$$

And so using formulas (1) and (6),

$$\begin{aligned}
\lim_{a\to 0^-} \frac{T}{\tilde{T}} &= \lim_{a\to 0^-} \frac{f(a)(f'(a)-f'(b))}{f'(a)f'(b)(b-a)} \\
&= \lim_{a\to 0^-} \frac{f(a)(f'(a)-f'(\gamma(a)))}{f'(a)f'(\gamma(a))(\gamma(a)-a)} \\
&= \lim_{a\to 0^-} \frac{f(a)(\gamma'(a)-1)}{f'(a)(\gamma(a)-a)} \\
&= \frac{1}{2n}. \qquad (8)
\end{aligned}$$

Putting this together with Theorem 1 and (7), we get the generalization to Archimedes' squaring of the parabola.

Theorem 2 *(Generalized Arcimedean Squaring) Given the hypotheses of Theorem 1 and the definition of $\tilde{T}$ above,*

$$\lim_{\substack{PQ\to 0 \\ \overline{PQ}||T_R\mathcal{C}}} \frac{A}{\tilde{T}} = \frac{2}{2n+1},$$

where the limit is taken over pairs of points $P, Q \in \mathcal{C}$, on opposite sides of $R \in \mathcal{C}$ and such that $\overline{PQ} \parallel T_R\mathcal{C}$.

It is necessary in the limit that $PQ \parallel T_R\mathcal{C}$ for otherwise the limit may not exist. Also, just as in quadrature, the ratio $A/\tilde{T}$ is constantly $2/(2n+1)$ if $\mathcal{C}$ is the graph of $f(x) = x^{2n}$.

The Two Triangle Theorem

We will now describe another area fact about parabolas that can be gleaned from Archimedes' Theorem. In Figure 4, the lines $\overline{PR'}$, $\overline{QR'}$ and $\overline{Q'P'}$ are all tangent to the parabolic arc at P, Q and R respectively. There is no assumption about R other than it lies between P and Q on the arc.

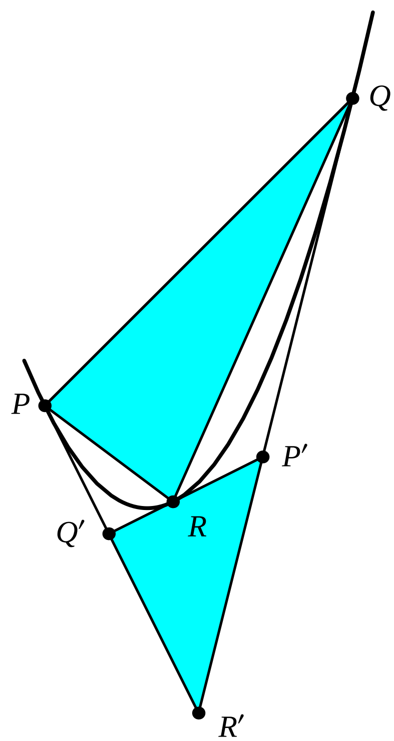

Figure 4: Two Triangle Theorem

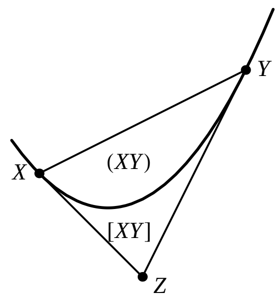

Figure 5: Defnitions of (XY) and $[XY]$

We will now show how Archimedes could have used his squaring of the parabola to prove

$$\frac{\text{Area } \triangle PQR}{\text{Area } \triangle P'Q'R'} = 2.$$

A simplified notation will help in the proof. For the parabolic arc and associated tangent lines pictured in Figure 5, (XY) and $[XY]$ will denote the indicated areas.

According to Archimedes' squaring of the parabola, (XY) is two-thirds of the area of $\triangle XYZ$, and so $(XY) = 2[XY]$. Thus

$$\begin{aligned}\frac{\text{Area } \triangle PQR}{\text{Area } \triangle P'Q'R'} &= \frac{(PQ) - (PR) - (QR)}{[PQ] - [PR] - [QR]} \\ &= \frac{2\left([PQ] - [PR] - [QR]\right)}{[PQ] - [PR] - [QR]} \\ &= 2.\end{aligned}$$

So the question is, to what extent does the above fact transfer to analytic plane curves? The answer is given by the next theorem. The theorem will refer to the same named points P, Q, R, P', Q', R' used above on an analytic plane curve $\mathcal{C}$. We will also let $T = \text{Area } \triangle PQR$ and $T' = \text{Area } \triangle P'Q'R'$.

Theorem 3 *(Two Triangle Theorem) Referring to the previous paragraph, if $R \in \mathcal{C}$ is a point of order 2, then*

$$\lim_{PQ \to 0} \frac{T}{T'} = 2,$$

where the limit is taken over pairs of points $P, Q \in \mathcal{C}$ on opposite sides of R. If $R \in \mathcal{C}$ is of order $2n$, $n > 1$, then

$$\lim_{\substack{PQ \to 0 \\ \overline{PQ} || T_R\mathcal{C}}} \frac{T}{T'} = \frac{2n}{(2n-1)^2},$$

where the limit is taken over points $P, Q \in \mathcal{C}$ that are on opposite sides of R and $\overline{PQ}$ is parallel to the tangent line $T_R\mathcal{C}$.

Before we set about on the proof of the Two Triangle Theorem, a few remarks are in order. The additional hypotheses in the second part of the theorem, $\overline{PQ}||T_R\mathcal{C}$ in the limit, is necessary, for otherwise the limit may not exist. Also, just as before, if $\mathcal{C}$ is the graph of $f(x) = x^{2n}$ and $\overline{PQ}||T_R\mathcal{C}$, then it is readily verified that $T/T' = 2n/(2n-1)^2$.

Proof. (Two Triangle Theorem) We will prove the second assertion first. Our starting point will be the proof of Theorem 1, and we will freely use the notation and definitions from that proof. It is fairly straightforward to derive the area enclosed by triangle $\triangle P'R'Q'$, assuming $f(a) = f(b)$ (you may use Figure 6 with $\overline{PQ}$ tilted so that it is parallel to the horizontal axis),

$$T' = \frac{\left(f(a)(f'(b) - f'(a)) + f'(a)f'(b)(b-a)\right)^2}{2f'(a)f'(b)\left(f'(a) - f'(b)\right)}. \tag{9}$$

Using this, along with formula (2), we get

$$\frac{T}{T'} = \frac{f(a)(f'(a) - f'(b))f'(a)f'(b)(b-a)}{\left(f(a)(f'(b) - f'(a)) + f'(a)f'(b)(b-a)\right)^2} \tag{10}$$

If we let $X = f(a)(f'(a) - f'(b))$ and $Y = f'(a)f'(b)(b-a)$, and use the fact $f(a) = f(b)$, then

$$\begin{aligned} \frac{T}{T'} &= \frac{XY}{(-X+Y)^2} \\ &= \frac{\frac{X}{Y}}{\left(-\frac{X}{Y}+1\right)^2}. \end{aligned} \tag{11}$$

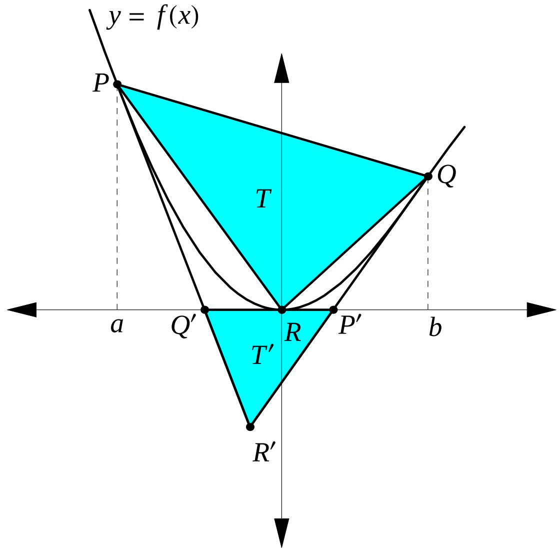

Figure 6: Proof of the Two Triangle Theorem

But

$$\begin{aligned}\frac{X}{Y} &= \frac{f(a)(f'(a)-f'(b))}{f'(a)f'(b)(b-a)} \\ &= \frac{T}{\tilde{T}} \\ &\to \frac{1}{2n}\end{aligned}$$

by formula (8). Substituting this into (11) proves the second part of the theorem,

$$\lim_{a\to 0^-} \frac{T}{T'} = \frac{2n}{(2n-1)^2}.$$

The proof of the first part will require a few preliminaries. We assume $f(x) = c_2x^2 + \cdots$ and $c_2 > 0$. (See Figure 6.)

Since we may no longer assume $f(a)$ and $f(b)$ are the same, we must use different formulas for T and T', derived from the standard vector formula for triangle area

$$\text{Area of } \triangle ABC = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||.$$

Both formulas are now straightforward calculations,

$$T = \frac{bf(a) - af(b)}{2} \tag{12}$$

and

$$T' = \frac{\left(f(a)f'(b) - f'(a)f(b) + f'(a)f'(b)(b-a)\right)^2}{2f'(a)f'(b)(f'(a) - f'(b))}.$$

So

$$\frac{T}{T'} = \frac{f'(a)f'(b)(bf(a) - af(b))(f'(a) - f'(b))}{\left(f(a)f'(b) - f'(a)f(b) + f'(a)f'(b)(b-a)\right)^2}.$$

To complete the proof, we will need four factorizations:

1. $f'(a) = a\varphi_1(a)$ where $\lim_{a\to 0}\varphi_1(a) = 2c_2$.

2. $f'(a) - f'(b) = (a-b)\varphi_2(a,b)$ where $\lim_{a,b\to 0}\varphi_2(a,b) = 2c_2$.

3. $bf(a) - af(b) = ab(a-b)\varphi_3(a,b)$ where $\lim_{a,b\to 0}\varphi_3(a,b) = c_2$.

4. $f'(a)f(b) - f(a)f'(b) = ab(b-a)\varphi_4(a,b)$ where $\lim_{a,b\to 0}\varphi_4(a,b) = 2c_2^2$.

Once we establish these factorizations the proof is complete,

$$\begin{aligned}
\lim_{a,b\to 0}\frac{T}{T'} &= \lim_{a,b\to 0}\frac{a\varphi_1(a)b\varphi_1(b)ab(a-b)\varphi_3(a,b)(a-b)\varphi_2(a,b)}{\left(ab(b-a)\varphi_4(a,b) - a\varphi_1(a)b\varphi_1(b)(b-a)\right)^2} \\
&= \lim_{a,b\to 0}\frac{\varphi_1(a)\varphi_1(b)\varphi_3(a,b)\varphi_2(a,b)}{\left(\varphi_4(a,b) - \varphi_1(a)\varphi_1(b)\right)^2} \\
&= \frac{8c_2^4}{\left(2c_2^2 - 4c_2^2\right)^2} \\
&= 2.
\end{aligned}$$

We will leave the proofs of the first three factorizations to the reader. They are straightforward infinite series manipulations. The fourth is a little more difficult and is given below. We start with

$$f(x) = \sum_{k=2}^{\infty} c_k x^k.$$

Then,

$$\begin{aligned}
f'(a)f(b) - f(a)f'(b) &= \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} kc_k c_m (a^{k-1}b^m - a^m b^{k-1}) \\
&= ab\sum_{k=2}^{\infty}\sum_{m=2}^{\infty} kc_k c_m (a^{k-2}b^{m-1} - a^{m-1}b^{k-2}) \\
&= ab\Big(\sum_{m-1<k-2} kc_k c_m a^{m-1}b^{m-1}(a^{k-m-1} - b^{k-m-1}) + \\
&\qquad \sum_{k-2<m-1} kc_k c_m a^{k-2}b^{k-2}(b^{m-k+1} - a^{m-k+1})\Big) \\
&= ab(b-a)\Big(-\sum_{m-1<k-2} kc_k c_m a^{m-1}b^{m-1}\sum_{j=0}^{k-m-2} b^j a^{k-m-2-j} + \\
&\qquad \sum_{k-2<m-1} kc_k c_m a^{k-2}b^{k-2}\sum_{j=0}^{m-k} a^j b^{m-k-j}\Big) \\
&= ab(b-a)\varphi_4(a,b),
\end{aligned}$$

where we have defined $\varphi_4(a,b)$ in the previous line. Then,

$$\lim_{a,b\to 0}\varphi_4(a,b) = \varphi_4(0,0)$$

and the only non-zero contribution to $\varphi_4(0,0)$ is from the second sum when $k=2$ and $m=2$, giving

$$\lim_{a,b\to 0}\varphi_4(a,b) = 2c_2^2.$$

This completes the proof of the Two Triangle Theorem.

New Directions

Archimedes' squaring of the parabola inspired the Two Triangle Theorem, which in turn bears a certain kinship to the so-called *osculating circle* of differential geometry. Let us recall the definition of the osculating circle. For any three points P, Q and R on the curve $\mathcal{C}$, we construct the circumcircle

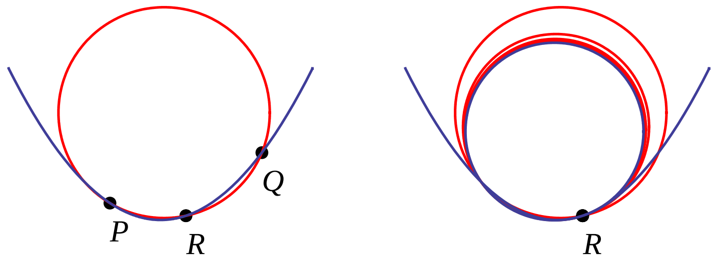

Figure 7: Constructing the Osculating Circle

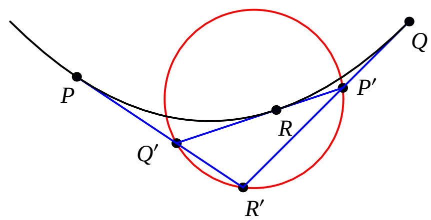

Figure 8: Circumcircle of $\triangle P'Q'R'$

of $\triangle PQR$. The osculating circle, $\mathcal{O}$, to $\mathcal{C}$ at R is the limit circle of these circumcircles as P and Q approach R along $\mathcal{C}$ and on opposite sides of R. (See Figure 7.) If the curvature to $\mathcal{C}$ at R, κ, is non-zero then the osculating circle exists and its radius is equal to $1/\kappa$, see [11].

Now it is perhaps natural to construct the circumcircles of the triangles $\triangle P'Q'R'$. (See Figure 8.) As $PQ \to 0$ and P and Q are on $\mathcal{C}$ and on opposite sides of R, it appears that we also get a sequence of circles that converges to a circle $\mathcal{O}'$ that we might call the *derived osculating circle*. The two limit circles at R, $\mathcal{O}$ and $\mathcal{O}'$, are pictured in Figure 9. The larger circle is the osculating circle $\mathcal{O}$.

If we let r and r' be the respective radii of $\mathcal{O}$ and $\mathcal{O}'$, then computer experiments suggest

$$\lim_{PQ\to 0} \frac{r}{r'} = 4,$$

so long as the point R is of order 2. However, if R is a point of order $2n$ with

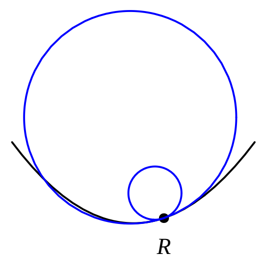

Figure 9: The osculating circle $\mathcal{O}$ (large) and derived osculating circle $\mathcal{O}'$ (small)

$n > 1$, then the above limit may not exist, so we restrict the limit:

$$\lim_{\substack{PQ \to 0 \\ \overline{PQ} || T_R\mathcal{C}}} \frac{r}{r'} = \frac{4n^2}{2n-1}$$

where the limit is taken over pairs of points $P, Q \in \mathcal{C}$ on opposite sides of R and so that $\overline{PQ} || T_R\mathcal{C}$.

There is clearly something of a general nature going on here. We suggest the following setting.

By a *triangle function* we will mean a real valued function $\mathcal{T}$ defined on triangles in the plane so that $\mathcal{T}(\triangle_1) = \mathcal{T}(\triangle_2)$ if $\triangle_1$ is congruent to $\triangle_2$. The area and circumradius functions from above are two examples. The general question is this. Using the points P, Q, R, P', Q', R' as we have been doing, R is a point of order 2, and assuming $\mathcal{T}$ is a triangle function, what is the value, if it exists, of

$$L = \lim_{PQ \to 0} \frac{\mathcal{T}(\triangle PQR)}{\mathcal{T}(\triangle P'Q'R')}?$$

And similarly, what about the limit

$$L_{||} = \lim_{\substack{PQ \to 0 \\ \overline{PQ} || T_R\mathcal{C}}} \frac{\mathcal{T}(\triangle PQR)}{\mathcal{T}(\triangle P'Q'R')}$$

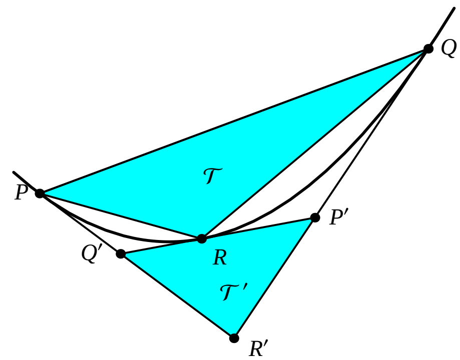

Figure 10: $\mathcal{T} = \mathcal{T}(\triangle PQR)$ and $\mathcal{T}' = \mathcal{T}(\triangle P'Q'R')$

if the order of R is $2n$ $(n > 1)$?

Some more analysis and computer experiments suggest there is no simple general answer to this question. Here are some results to ponder.

1. As we saw earlier, if $\mathcal{T}(\triangle) = \text{area}(\triangle)$, then $L = 2$ and $L_{||} = 2n/(2n-1)^2$.

2. Experimentally, $\mathcal{T}(\triangle) = \text{perimeter}(\triangle)$, then $L = 2$ and $L_{||} = 2n/(2n-1)$.

3. But if $\mathcal{T}(\triangle) = c + \tau(\triangle)$, where c is a fixed non-zero number and τ is either area or perimeter , then $L = L_{||} = 1$.

4. And as we saw above, if $\mathcal{T}(\triangle) = \text{circumradius}(\triangle)$, then $L = 4$ and $L_{||} = 4n^2/(2n-1)$. But if $\mathcal{T}(\triangle) = c + \text{circumradius}(\triangle)$ where c is a constant, then $L = (4\kappa c + 4)/(4\kappa c + 1)$ where κ is the curvature to $\mathcal{C}$ at R. On the other hand, $L_{||} = 4n^2/(2n-1)$ even if $c \neq 0$.

5. If $\mathcal{T}(\triangle) = \text{inradius}(\triangle)$, then computer experiments suggest $L = 1$ and $L_{||} = 1/(2n-1)$.

6. Also experimentally, if $\mathcal{T}(\triangle)$ is the cube root of the product of the three side lengths of $\triangle$, then $L = 2$ and $L_{||} = 2n/(2n-1)$.

Eureka, anyone?

Acknowledgment I would like to thank the anonymous referees for their insightful and helpful suggestions that greatly improved the exposition of the ideas presented in this article.

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