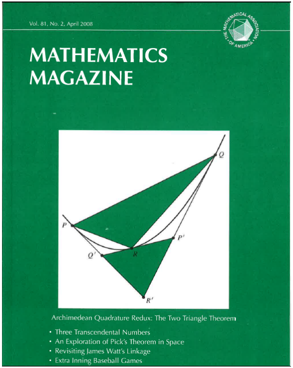

Archimedean Quadrature Redux

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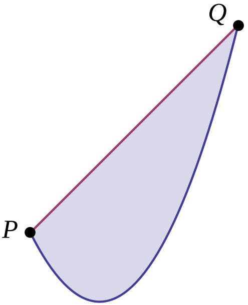
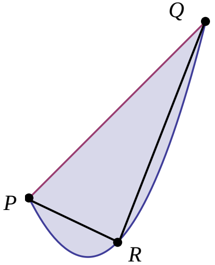
Archimedes' Quadrature of the Parabola

Problem: Measure the parabolic area.



Archimedes' Quadrature of the Parabola

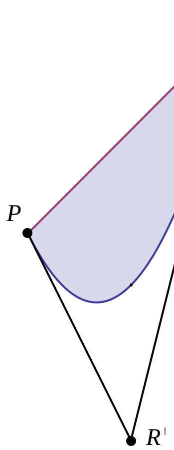
Archimedes' Solution: Locate point R on arc with maximum vertical distance from $\overline{PQ}$. (Turns out, the tangent line to the arc at R is parallel to $\overline{PQ}$.)



(AQP) Parabolic (shaded) Area $= \dfrac{4}{3}(\triangle PQR)$.

Archimedes' Squaring of the Parabola

Archimedes' other solution: Let $\triangle PQR'$ be the so-called Archimedean Triangle, where $\overline{PR'}$ and $\overline{QR'}$ are respective tangents.

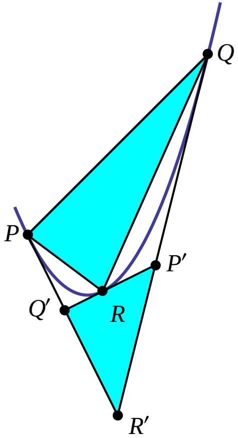


(ASP) Parabolic (shaded) Area $= \dfrac{2}{3}(\triangle PQR')$.

Two Triangles Theorem

Calculus Problem. For the parabolic arc with respective tangent lines pictured, compute the ratio of areas

$$\frac{\triangle PQR}{\triangle P'Q'R'}.$$



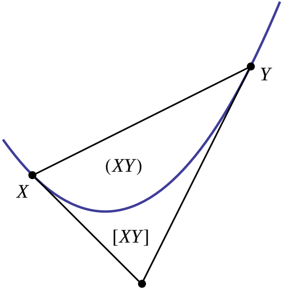
Answer. (TTT)

$$\frac{\triangle PQR}{\triangle P'Q'R'} = 2.$$

Two Triangles Theorem

TTT is a consequence of ASP.

Define areas (XY) and $[XY]$ as pictured.

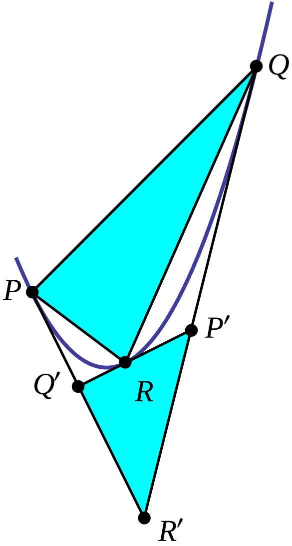


Then,

$$\textbf{ASP} \implies (XY) = \frac{2}{3}((XY) + [XY])$$
$$\implies (XY) = 2[XY].$$

Two Triangles Theorem

TTT is a consequence of ASP.



$$\frac{\triangle PQR}{\triangle P'Q'R'} = \frac{(PQ)-(PR)-(QR)}{[PQ]-[PR]-[QR]}$$

$$= \frac{2([PQ]-[PR]-[QR])}{[PQ]-[PR]-[QR]}$$

$$= 2.$$

Generalizing...

New Question:. What happens when the curve is no longer a parabola?

Reasonable Restrictions? How about polynomial curves? Rational curves? Analytic curves?

Definition. A curve $\mathcal{C}$ will be called *analytic of order n* at a point $R \in \mathcal{C}$ if there is a coordinate system at R with the two respective axes tangent and normal to $\mathcal{C}$ at R so that in a neighborhood of R, $\mathcal{C}$ is the graph of an analytic function (power series)

$$f(x) = c_n x^n + c_{n+1} x^{n+1} + \cdots ,$$

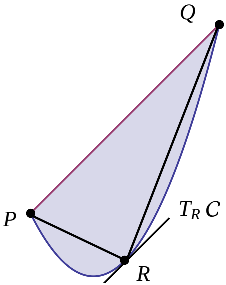
where $c_n \neq 0$. For our purposes, n will always be an even positive integer.

Note. A point R on a curve $\mathcal{C}$ is of order 2 precisely when the curvature of $\mathcal{C}$ is non-zero at R.

Generalized Archimedean Quadrature

Fix $R \in \mathcal{C}$ and pick points $P, Q \in \mathcal{C}$ on opposite sides of R and so that $\overline{PQ}$ is parallel to the tangent line to $\mathcal{C}$ at R. Then, let P and Q approach R along $\mathcal{C}$.

GAQ. Assume $\mathcal{C}$ is an analytic plane curve, and $R \in \mathcal{C}$ is a point of order $2n$, $n \geq 1$.

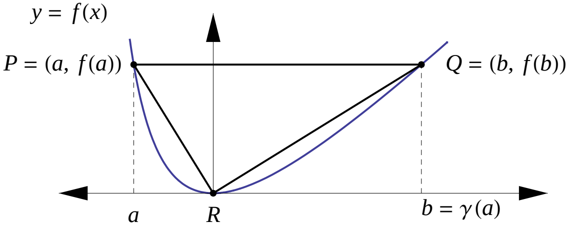


Then,

$$\lim \frac{(PQ)}{\triangle PQR} = \frac{4n}{2n+1}.$$

Proof of GAQ

Proof.



$$\lim \frac{(PQ)}{\triangle PQR} = \lim_{a\to 0} \frac{2\left(f(a)(\gamma(a)-a) - \int_a^{\gamma(a)} f(x)\, dx\right)}{f(a)(\gamma(a)-a)},$$

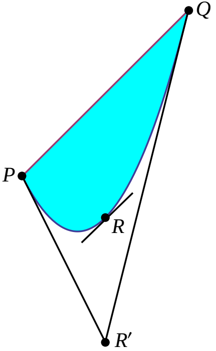
Use algebra, L'Hospital's Rule (several times), the Fundamental Theorem of Calculus & the Inverse Function Theorem. . . to get

$$\lim \frac{(PQ)}{\triangle PQR} = \frac{4n}{2n+1}.$$

Generalized Archimedean Squaring

Fix $R \in \mathcal{C}$ and pick points $P, Q \in \mathcal{C}$ on opposite sides of R and so that $\overline{PQ}$ is parallel to the tangent line to $\mathcal{C}$ at R. Let R' be the intersection of the tangents to $\mathcal{C}$ at the points P and Q, respectively. Then, let P and Q approach R along $\mathcal{C}$.

GAS. Assume $\mathcal{C}$ is an analytic plane curve, and $R \in \mathcal{C}$ is a point of order $2n$, $n \geq 1$.

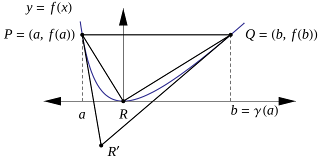


Then,

$$\lim \frac{(PQ)}{\triangle PQR'} = \frac{2}{2n+1}.$$

Proof of GAS

Proof.



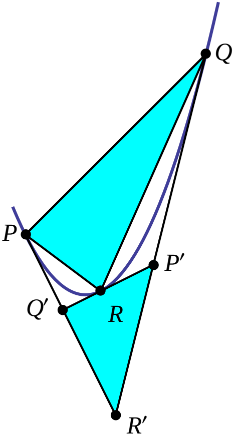
$$\lim \frac{(PQ)}{\triangle PQR'} = \lim \frac{(PQ)}{\triangle PQR} \cdot \frac{\triangle PQR}{\triangle PQR'}$$

$$= \frac{4n}{2n+1} \cdot \lim_{a \to 0} \frac{f(a)(f'(a) - f'(\gamma(a)))}{f'(a)f'(\gamma(a))(\gamma(a) - a)},$$

Use algebra, L'Hospital's Rule (several times), the Fundamental Theorem of Calculus & the Inverse Function Theorem… to get

$$\lim \frac{(PQ)}{\triangle PQR'} = \frac{4n}{2n+1} \cdot \frac{1}{2n} = \frac{2}{2n+1}.$$

Recall Two Triangles Theorem



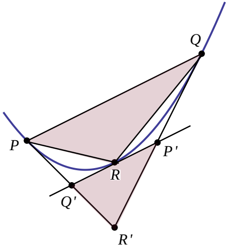
TTT for Parabolas

$$\frac{\triangle PQR}{\triangle P'Q'R'} = 2.$$

Generalized Two Triangles Theorem–Parallel Case

Fix $R \in \mathcal{C}$ and pick points $P, Q \in \mathcal{C}$ on opposite sides of R and so that $\overline{PQ}$ is parallel to the tangent line to $\mathcal{C}$ at R. Let P', Q', R' be the respective intersection points of the pictured tangents to $\mathcal{C}$. Then, let P and Q approach R along $\mathcal{C}$.

GTTT–Parallel. Assume $\mathcal{C}$ is an analytic plane curve, and $R \in \mathcal{C}$ is a point of order $2n$, $n \geq 1$ and $\overleftrightarrow{PQ} || \overleftrightarrow{P'Q'}$.



Then,

$$\lim \frac{\triangle PQR}{\triangle P'Q'R'} = \frac{2n}{(2n-1)^2}.$$

Proof.

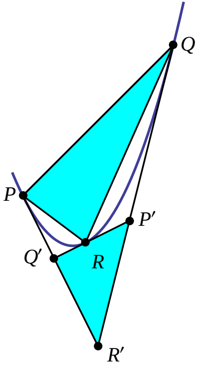
Similar to GAQ & GAS.

□

Generalized Two Triangles Theorem–Non-Parallel Case

Fix $R \in \mathcal{C}$ and pick points $P, Q \in \mathcal{C}$ on opposite sides of R. (No parallel requirement.) Let P', Q', R' be the respective intersection points of the pictured tangents to $\mathcal{C}$. Then, let P and Q approach R along $\mathcal{C}$.

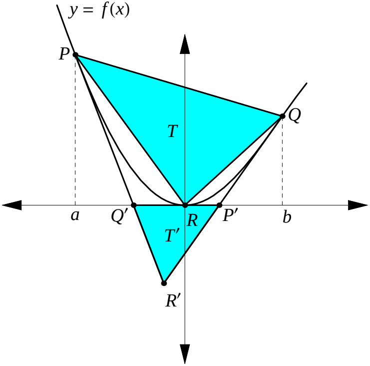
GTTT–Non-Parallel. Assume $\mathcal{C}$ is an analytic plane curve, and $R \in \mathcal{C}$ is a point of order 2. (So curvature at R is not zero.) No parallel assumption regarding $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$.



Then,

$$\lim \frac{\triangle PQR}{\triangle P'Q'R'} = 2.$$

Proof of TTT–Non-Zero Curvature Case



$$\frac{\triangle PQR}{\triangle P'Q'R'} = \frac{T}{T'} =$$

$$\frac{f'(a)f'(b)(bf(a) - af(b))(f'(a) - f'(b))}{(f(a)f'(b) - f'(a)f(b) + f'(a)f'(b)(b-a))^2}$$

Proof of TTT–Non-Zero Curvature Case

$$\frac{T}{T'} = \frac{f'(a)f'(b)(bf(a) - af(b))(f'(a) - f'(b))}{(f(a)f'(b) - f'(a)f(b) + f'(a)f'(b)(b-a))^2}$$

Factorizations (Proved using series manipulations.)

1. $f'(a) = a\varphi_1(a)$ where $\lim_{a\to 0} \varphi_1(a) = 2c_2$.
2. $f'(a) - f'(b) = (a-b)\varphi_2(a,b)$ where $\lim_{a,b\to 0} \varphi_2(a,b) = 2c_2$.
3. $bf(a) - af(b) = ab(a-b)\varphi_3(a,b)$ where $\lim_{a,b\to 0} \varphi_3(a,b) = c_2$.
4. $f'(a)f(b) - f(a)f'(b) = ab(b-a)\varphi_4(a,b)$ where $\lim_{a,b\to 0} \varphi_4(a,b) = 2c_2^2$.

Proof of TTT–Non-Zero Curvature Case

Using the factorizations. . .

$$
\begin{aligned}
\lim_{a,b\to 0} \frac{T}{T'} &= \lim_{a,b\to 0} \frac{f'(a)f'(b)(bf(a)-af(b))(f'(a)-f'(b))}{(f(a)f'(b)-f'(a)f(b)+f'(a)f'(b)(b-a))^2} \\
&= \lim_{a,b\to 0} \frac{a\varphi_1(a)b\varphi_1(b)ab(a-b)\varphi_3(a,b)(a-b)\varphi_2(a,b)}{(ab(b-a)\varphi_4(a,b)-a\varphi_1(a)b\varphi_1(b)(b-a))^2} \\
&= \lim_{a,b\to 0} \frac{\varphi_1(a)\varphi_1(b)\varphi_3(a,b)\varphi_2(a,b)}{(\varphi_4(a,b)-\varphi_1(a)\varphi_1(b))^2} \\
&= \frac{8c_2^4}{\left(2c_2^2-4c_2^2\right)^2} \\
&= 2.
\end{aligned}
$$

End of Proof

New Direction–Triangle Functions

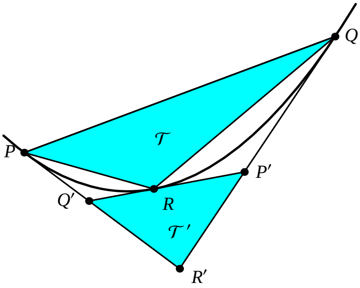
A *triangle function* is a real valued function $\mathcal{T}$ defined on triangles in the plane so that $\mathcal{T}(\triangle_1) = \mathcal{T}(\triangle_2)$ if $\triangle_1 \cong \triangle_2$.

Examples.

- $\mathcal{T}(\triangle) =$ Area, or perimeter, enclosed by $\triangle$.
- $\mathcal{T}(\triangle) =$ in-radius of $\triangle$.
- $\mathcal{T}(\triangle) =$ circum-radius of $\triangle$. Note: $\lim \mathcal{T}(\triangle) =$ (radius of osculating circle) $= 1/$curvature.
- $\mathcal{T}(\triangle) =$ Length of the Euler line segment of $\triangle$.
- $\mathcal{T}(\triangle) =$ Area, or perimeter, of Morley's miracle equilateral triangle in $\triangle$.

New Direction–Triangle Functions

Notation and conventions as above. Define. . .if the limits exist



- If $R \in \mathcal{C}$ is an order 2 point,

$$L = \lim \frac{\mathcal{T}(\triangle PQR)}{\mathcal{T}(\triangle P'Q'R')}$$

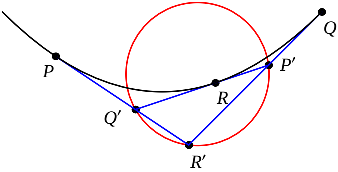
- If $R \in \mathcal{C}$ has order $2n$, $n \geq 1$,

$$L_{||} = \lim_{\overline{PQ}||T_R\mathcal{C}} \frac{\mathcal{T}(\triangle PQR)}{\mathcal{T}(\triangle P'Q'R')}$$

Results of Computer Experiments–Conjectures

1. If $\mathcal{T}(\triangle) = \text{Perimeter}(\triangle)$, then $L = 2$ and $L_{||} = 2n/(2n-1)$. (Same result as $\mathcal{T}(\triangle) = \text{Area}(\triangle)$.)
2. But if $\mathcal{T}(\triangle) = c + \tau(\triangle)$, where c is a fixed non-zero number and τ is either area or perimeter , then $L = L_{||} = 1$.
3. If $\mathcal{T}(\triangle) = \text{Circumradius}(\triangle)$, then $L = 4$ and $L_{||} = 4n^2/(2n-1)$. But if $\mathcal{T}(\triangle) = c + \text{Circumradius}(\triangle)$ where c is a constant, then $L = (4\kappa c + 4)/(4\kappa c + 1)$ where κ is the curvature to $\mathcal{C}$ at R. On the other hand, $L_{||} = 4n^2/(2n-1)$ even if $c \neq 0$.
4. If $\mathcal{T}(\triangle) = \text{inradius}(\triangle)$, then $L = 1$ and $L_{||} = 1/(2n-1)$.
5. If $\mathcal{T}(\triangle)$ is the cube root of the product of the three side lengths of $\triangle$, then $L = 2$ and $L_{||} = 2n/(2n-1)$. (Same as for perimeter and area.)

Thanks!



Eureka?