Knot Quandles & Quandle Knots

Larry W. Cusick

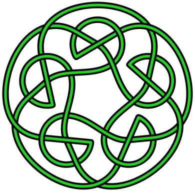
California State University, Fresno

October 13, 2012

Knots & Labeling

What is a Knot?

Definition. A *knot* is a simple closed polygonal curve in $\mathbf{R}^3$.

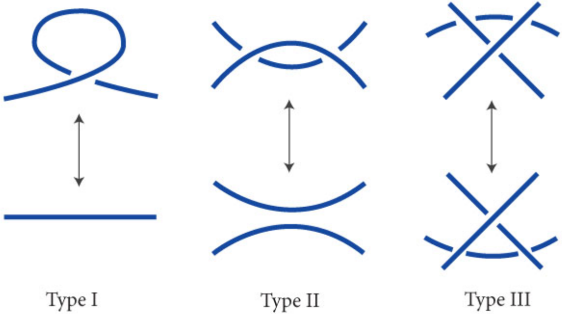


Think of the knot as made up from a large (finite) number of straight-line segments. We will work with their *diagrams*: projections onto a plane with the strand crossings indicated–as pictured.

Links

Links are interlocking knots. We will use the term knots to include knots and links.

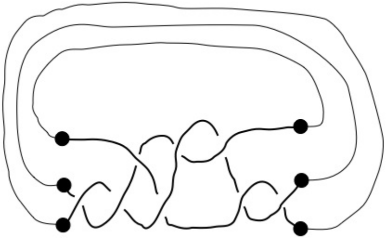
Reidemeister Moves



Working Definition. Two knots are equivalent if, and only if, one knot diagram can be gotten from the other by a finite number of Reidemeister moves.

Braid Representation of a Knot

A braid gives rise to a knot/link.

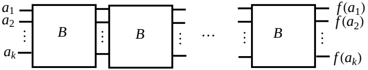


Theorem (Markov)

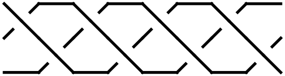
Every knot/link has a braid representation.

Knots/Links Wholesale

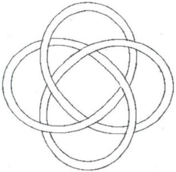
Start with a braid B. Link n copies of B with itself to create a larger braid. Use the larger braid to form a knot/link by connecting left ends to corresponding right ends.



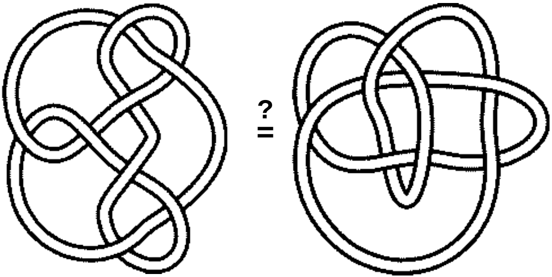
Example Torus knots come this way.



Connect the ends to get. . .

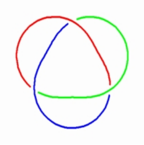


Distinguishing Knots



Why are knots 8_{15} and 8_{21} not equivalent?

Ralph Fox and Knot Coloring



3-Coloring Rules for Knot Diagrams

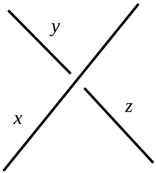
1. Each strand is assigned a color (R,G, B).
2. Use all three colors.
3. If there are two colors at a crossing, all three colors must appear at the crossing.

Theorem

If a knot diagram can be 3-colored, then any equivalent diagram of the knot can be 3-colored.

Consequence. The trefoil knot is not equivalent to the unknot.

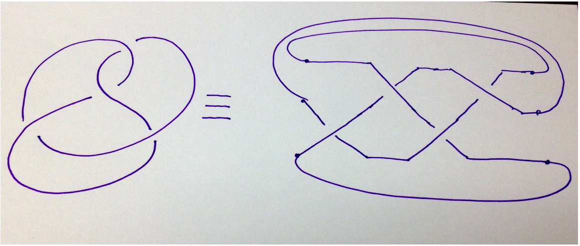
Generalized Coloring: Labeling Mod p



$\mathbf{Z}_p$-Labeling Rules for Knot Diagrams (p is a prime)

1. Each strand is labeled with an element from $\mathbf{Z}_p = \{0, 1, \dots, p-1\}$.
2. Use at least two of the elements of $\mathbf{Z}_p$.
3. If x labels the over-crossing and y and z label the under-crossings, then $z \equiv 2x - y \bmod p$.

Figure 8 Knot Labeling



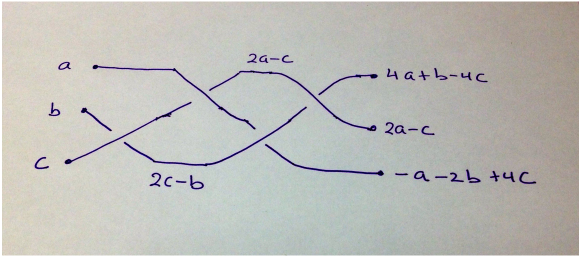
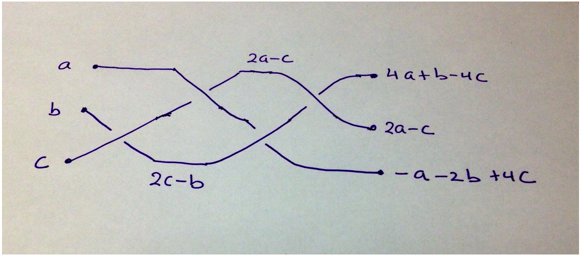


Figure 8 Solution

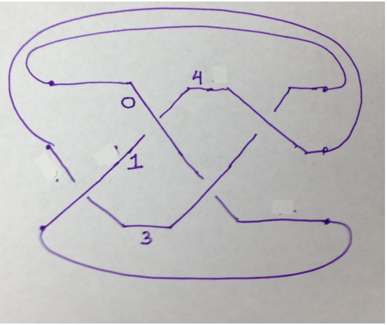


All equivalences mod p.

$$\begin{aligned} a &\equiv 4a + b - 4c \\ b &\equiv 2a - c \\ c &\equiv -a - 2b + 4c. \end{aligned}$$

Non-trivial solution when $p = 5$, where a and c satisfy $a \neq c$, $5a \equiv 5c \bmod p$ and $b \equiv -3a + 4c \bmod p$. (For instance, $(a, b, c) \equiv (0, 4, 1) \bmod 5$.

Figure 8 Knot Labeled



Theorem

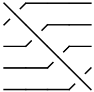
The number of $\mathbf{Z}_p$ labelings of a knot is an invariant of the equivalence class of the knot.

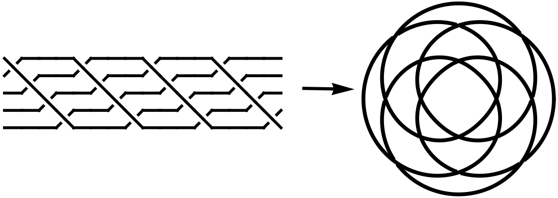
Torus Knots

A *Torus knot* is a knot that can be placed on an unknotted torus without self-intersections. All torus knots are gotten from iterating a braid B_r, q times. The resulting knot is $T(r, q)$.

Example. The torus knot $T(5, 4)$ is obtained by iterating the braid B_5, four times.

B_5 :





$\mathbf{Z}_p$-Labeling Results by CSUF Students

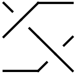
Theorem

(J. Bryan) The $T(r,q)$ Torus Knot can be $\mathbf{Z}_p$-labeled if, and only if, either

1. $r \equiv 0 \bmod 2$ *and* $q \equiv 0 \bmod p$ *or*
2. $r \equiv 0 \bmod p$ *and* $q \equiv 0 \bmod 2$

Theorem

(M. Rodriguez, A. Tibebu, I. Perez) Let B the the braid

$B:$ 

Then, the knot determined by the iterated braid B^n

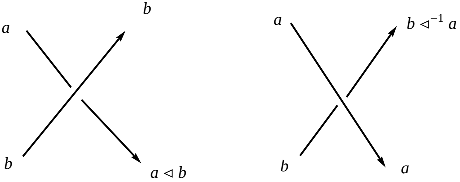
can be $\mathbf{Z}_p$ labeled if, and only if, p divides $5F_n$ if n is even and p divides L_n if n is odd, where F_n and L_n are the respective Fibonacci and Lucas numbers.

Knot Quandles

Knot Quandles

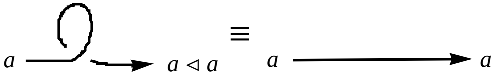
Conway & Wraith

Knot quandles generalize $\mathbf{Z}_p$ labelings. We start with a set K, to label the diagram, and two binary operations on K, $\triangleleft$ and $\triangleleft^{-1}$. The *Two Labeling Rules* are pictured below.

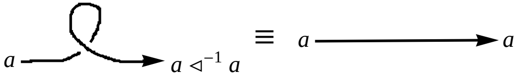


Compatibility Type I

The binary operations are designed so that the labeling is compatible with the (oriented) Reidemeister moves.

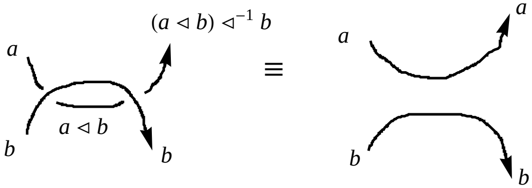


Thus, we need for all $a \in K$, $a \triangleleft a = a$.

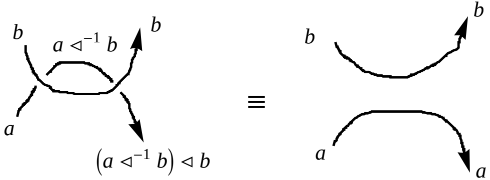


This gives us $a \in K$, $a \triangleleft^{-1} a = a$.

Compatibility Type II

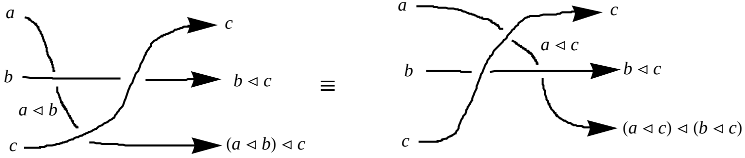


This gives us the identity for all $a, b \in K$, $(a \triangleleft b) \triangleleft^{-1} b = a$.

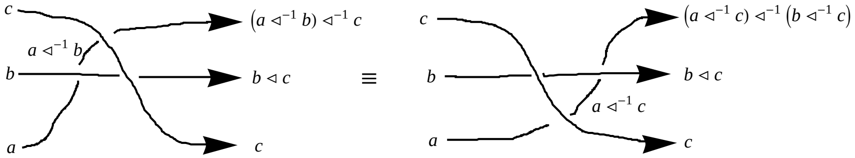


So, for all $a, b \in K$, $(a \triangleleft^{-1} b) \triangleleft b = a$.

Compatibility Type III



Giving us for all $a, b, c \in K$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$



The inverse version: $a, b, c \in K$,
$(a \triangleleft^{-1} b) \triangleleft^{-1} c = (a \triangleleft^{-1} c) \triangleleft^{-1} (b \triangleleft^{-1} c)$

Quandle Axioms

David Joyce

It turns out that we need only four of the above formulas–the other two are consequences of the axioms.

Definiton A *quandle* $(K, \triangleleft, \triangleleft^{-1})$ is a set K with two binary operations $\triangleleft$ and $\triangleleft^{-1}$ that satisfy

1. $\forall a \in K$,
   $a \triangleleft a = a$
2. $\forall a, b \in K$,
   2.1 $(a \triangleleft b) \triangleleft^{-1} b = a$ and
   2.2 $(a \triangleleft^{-1} b) \triangleleft b = a$
3. $\forall a, b, c \in K$,
   $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

Example of Quandles–$\mathbf{Z}_{n,q}$

For any unit q in a commutative ring K, we can define a quandle structure on K, called K_q, by

1. $a \triangleleft b = qa + (1-q)b$ and
2. $a \triangleleft^{-1} b = q^{-1}a + (1-q^{-1})b$.

If $K = \mathbf{Z}_n$, then we denote this quandle by $\mathbf{Z}_{n,q}$, where q is a unit in $\mathbf{Z}_n$. The classical $\mathbf{Z}_n$-labeling of a knot corresponds to the quandle $\mathbf{Z}_{n,-1} = \mathbf{Z}_{n,n-1}$.

Table for $\mathbf{Z}_{4,3}$:

$\triangleleft$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

Example of Quandles–Group Conjugation

Example 2. For a group G, and integer q, we can define a quandle structure on G by

1. $a \triangleleft b = b^q a b^{-q}$
2. $a \triangleleft^{-1} b = b^{-q} a b^q$

For example, the dihedral group D_3 has a presentation

$$D_3 =< a, b \mid a^3 = b^2 = 1, ab = ba^2 >= \{1, a, a^2, b, ab, a^2 b\}.$$

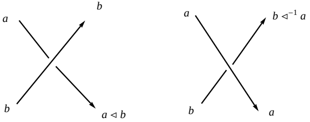
From this, we can construct the group quandle table for $q = 1$,

$$x \triangleleft y = yxy^{-1}.$$

$\triangleleft$	1	a	a^2	b	ab	a^2b
1	1	1	1	1	1	1
a	a	a	a	a^2	a^2	a^2
a^2	a^2	a^2	a^2	a	a	a
b	b	a^2b	ab	b	a^2b	ab
ab	ab	b	a^2b	a^2b	ab	b
a^2b	a^2b	ab	b	ab	b	a^2b

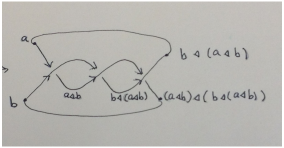
Labeling Knots with Quandles

Definition. A Labeling of a diagram of a knot X by a quandle K was a labeling of the strands in the diagram with at least two elements from K so that the two labeling rules are satisified.



Trefoil Example

Example. The trefoil knot braid representation:



This gives us two equations:

$$\begin{aligned} a &= b \triangleleft (a \triangleleft b) \\ b &= (a \triangleleft b) \triangleleft (b \triangleleft (a \triangleleft (b)). \end{aligned}$$

In the case that $K = \mathbf{Z}_{n,q}$, these equations are equivalent to a single equation

$$(q^2 - q + 1)a \equiv (q^2 - q + 1)b \bmod n.$$

Thus, the trefoill knot has a $\mathbf{Z}_{n,q}$ labeling if, and only if, $\gcd(q^2 - q + 1, n) \neq 1$. (Like $\mathbf{Z}_{7,3}$.)

Trefoil Example Continued

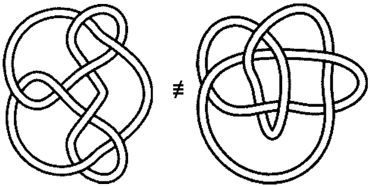
$$\begin{aligned} a &= b \triangleleft (a \triangleleft b) \\ b &= (a \triangleleft b) \triangleleft (b \triangleleft (a \triangleleft (b)). \end{aligned}$$

In the case that K is a group quandle G with $a \triangleleft b = b^q a b^{-q}$ for some $q \in \mathbf{Z}$, the two equations become equations in a group,

$$\begin{aligned} a &= b^q aba^{-1} b^{-q} \\ b &= b^q abab^{-1} a^{-1} b^{-q}. \end{aligned}$$

This is equivalent to the single group equation $aba = bab$. This has a solution in the quandle of transpositions $K = \{(12), (23), (13)\} \subset S_3$ with $q = 1$, where $a = (12)$ and $b = (23)$.

Back to 8_{15} & 8_{21}



The knots 8_{15} and 8_{21} are not equivalent because 8_{15} can be labeled by the quandle $\mathbf{Z}_{10,3}$ but not $\mathbf{Z}_{10,9}$, whereas 8_{21} can be labeled by the quandle $\mathbf{Z}_{10,9}$ but not $\mathbf{Z}_{10,3}$.

All knots with 12 or fewer crossings (nearly 3000 knots) can be distinguished using only 20 quandles.

Knot Quandle Questions

1. Extend Bryan's result to the quandle $\mathbf{Z}_{n,q}$ for torus knots.
2. Extend Rodriguez, et al's result to the quandle $\mathbf{Z}_{n,q}$ for a simple iterated braid knot.
3. Do the above for more complicated braid knots.
4. Do the above for group quandles, $a \triangleleft b = b^q a b^{-1q}$ for small finite groups (like transpostion quandles).
5. J. Bryan proved a connected sum of knots can be labeled by $\mathbf{Z}_n$ if, and only if, one of its summands can be labeled. Is this true for $\mathbf{Z}_{n,q}$ quandles? What about more general quandles?

Quandle Knots

Quandle Knots

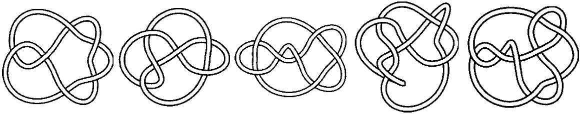
Definition. A knot X supports a quandle K if X can be labeled by K.

Question. Given a quandle K, which knots X support K?

More Reasonable Question. Given a quandle K and a simple braid B, which iterated braid knots B^n support K? Start with $K = \mathbf{Z}_{n,q}$.

Quandle Knot Example

Among the 35 knots of 8 or fewer crossings, 5 support the quandles $\mathbf{Z}_{7,2}$ and $\mathbf{Z}_{7,4}$:



Among these same 35 knots, none support the quandle $\mathbf{Z}_{4,3}$. However, this quandle *is* supported by the Torus link $T(3,3)$, as well as by the links B^{3k} where B is

B :

Knot-Equivalent Quandles

Experimental evidence suggests the two quandles $\mathbf{Z}_{7,2}$ and $\mathbf{Z}_{7,4}$ are supported by the same knots.

Definition. Quandles K_1 and K_2 are called *knot-equivalent* if they are supported by the same knots.

Questions.

- Are $\mathbf{Z}_{7,2}$ and $\mathbf{Z}_{7,4}$ knot-equivalent?
- Find other knot-equivalent pairs.
- Is there better terminology than "knot-equivalent?"
- Can it be proved that the quandle $\mathbf{Z}_{4,3}$ is supported only by links, not knots?

Isomorphic Quandles?

Example. The quandles $\mathbf{Z}_{7,4}$ and $\mathbf{Z}_{7,6}$ have quandle tables whose column cycle-structure are the same. Are they isomorphic?

Answer. No. They are distinguished by their quandle knots: $6_1, 6_3, 7_5$ in the case of $\mathbf{Z}_{7,4}$ and $5_2, 7_1, 7_7, 8_5$ in the case of $\mathbf{Z}_{7,6}$.

Quandle Theory

What would a Quandle Theory look like?

- Classify quandles into families?
- Structure of quandle tables?
- How to prove two quandles are not isomorphic?
- Quotient quandles by sub-quandles?
- What group theory ideas transfer to quandles?

Thanks