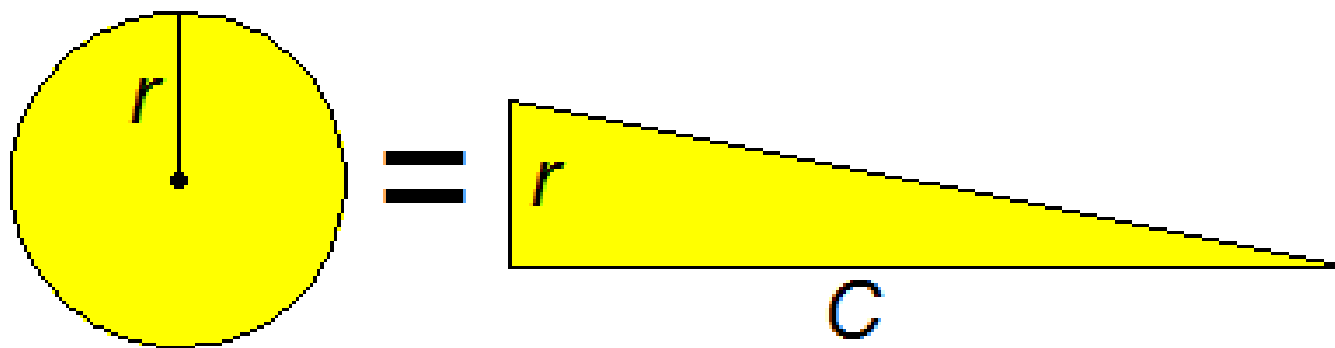
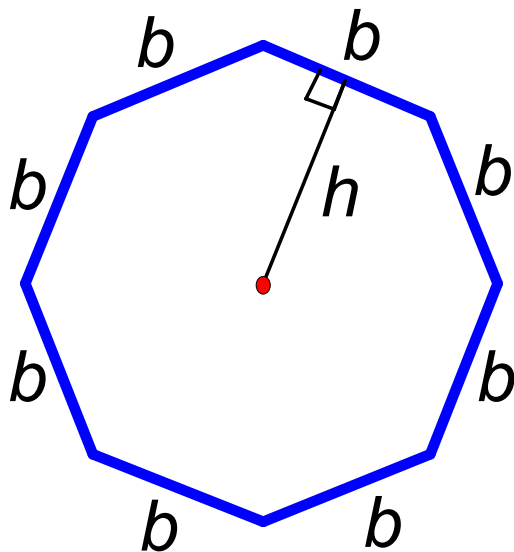


Archimedes (287 – 212 B.C.E.), in his treatise titled *Measurement of a Circle*, proved that the area of a circle having radius  $r$  is equal to the area of a right triangle having base  $C$  and height  $r$  where  $C$  is the circumference of the circle.



**Theorem:** The area of a regular polygon is  $\frac{1}{2}hQ$  where  $h$  is the perpendicular distance from the center to a side and  $Q$  is the perimeter of the polygon.



Octagon

$$\text{Perimeter } Q = 8b$$

$$\text{Area } A = (hQ)/2 = 4hb$$

## Proof:

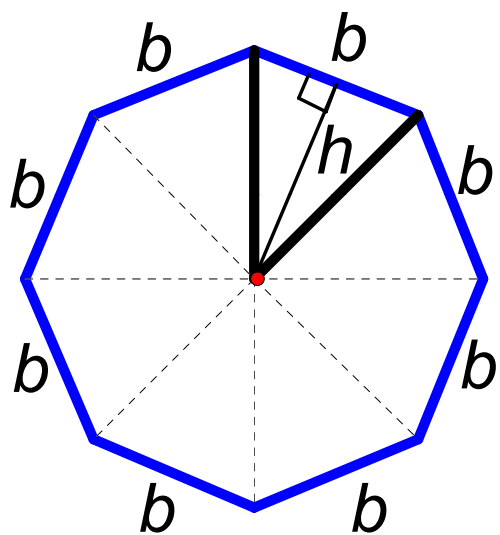
Area of triangle =  $\frac{1}{2}bh$ .

A regular  $n$ -gon has  $n$  sides, and thus  $n$  such triangles.

So the area of a regular  $n$ -gon is  $n(\frac{1}{2}bh)$ . Thus

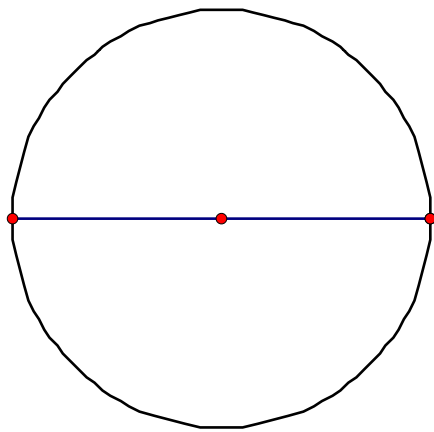
$$A = \frac{1}{2}h(nb) = \frac{1}{2}hQ$$

where  $Q$  is the perimeter of the regular  $n$ -gon.

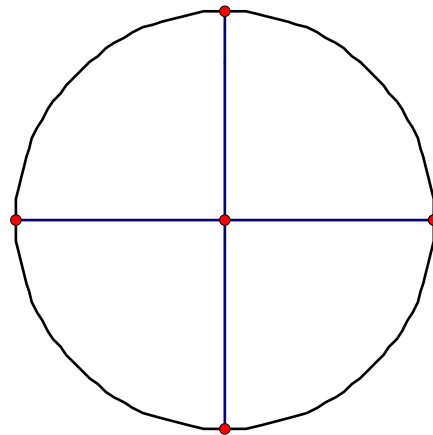


**Theorem:** A regular  $2^n$ -gon,  $n \geq 2$ , can be inscribed in a circle.

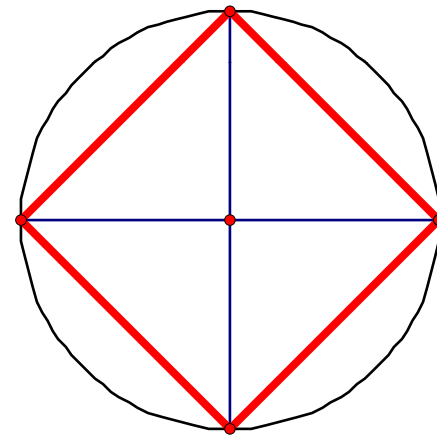
- Inscribing a square ( $2^2$ -gon) in a circle:



Construct a diagonal.

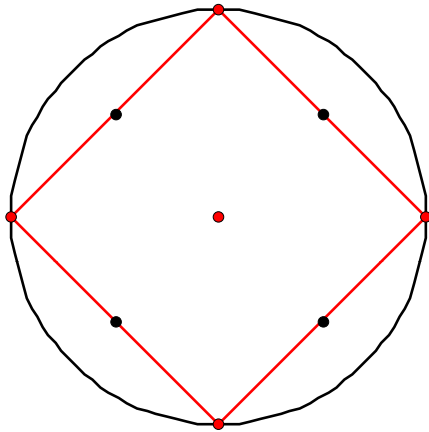


Construct a  $\perp$  to the diagonal.

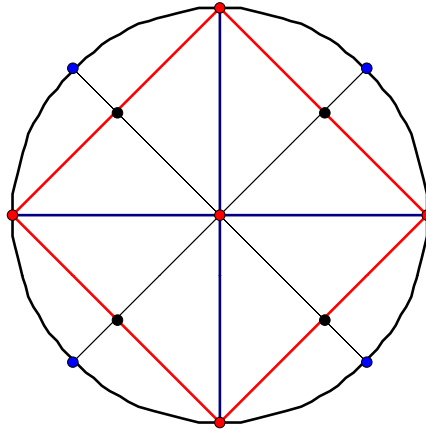


Connect the vertices.

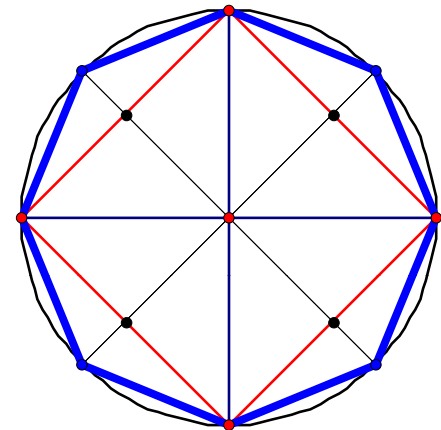
- Inscribing an octagon ( $2^3$ -gon) in a circle:



Bisect sides  
of inscribed  
square.

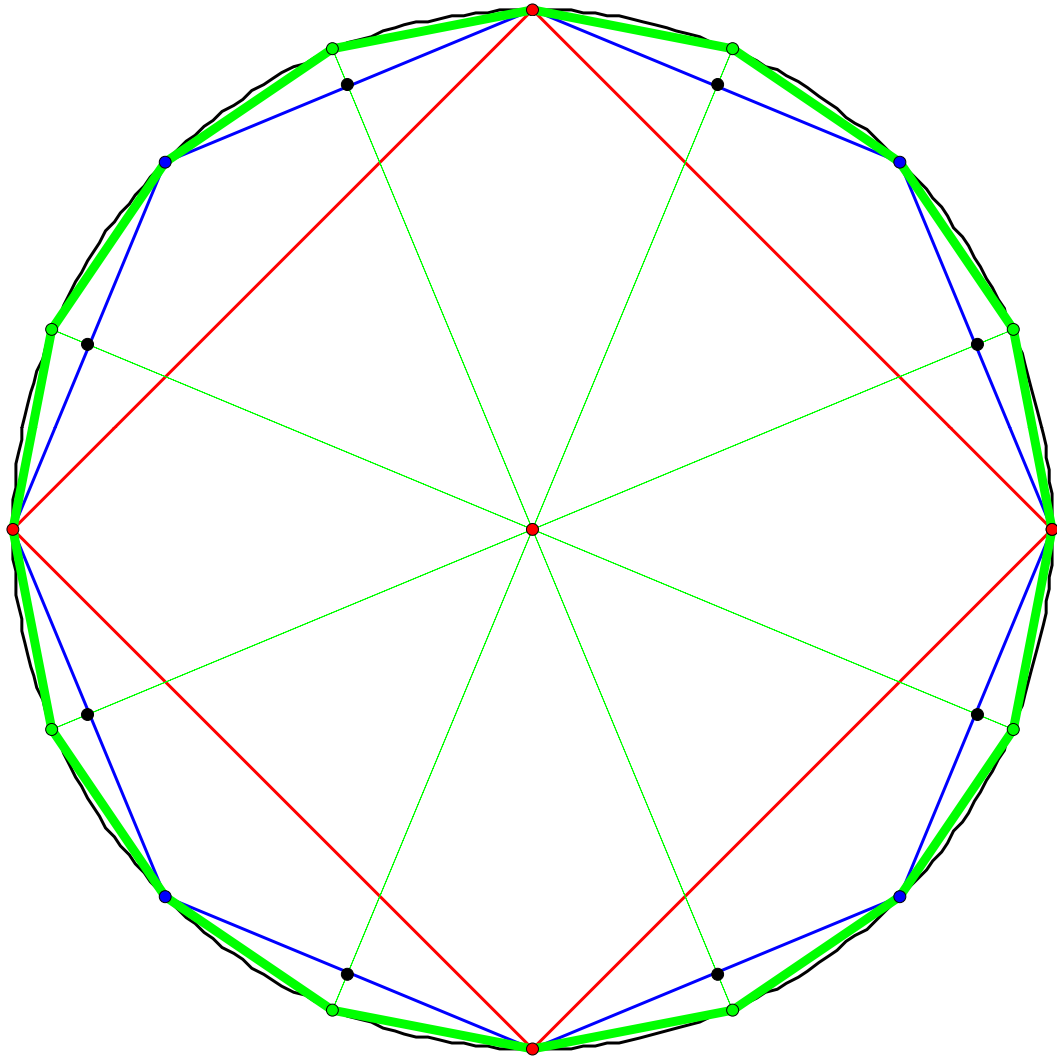


Construct radii  
through the  
midpoints.



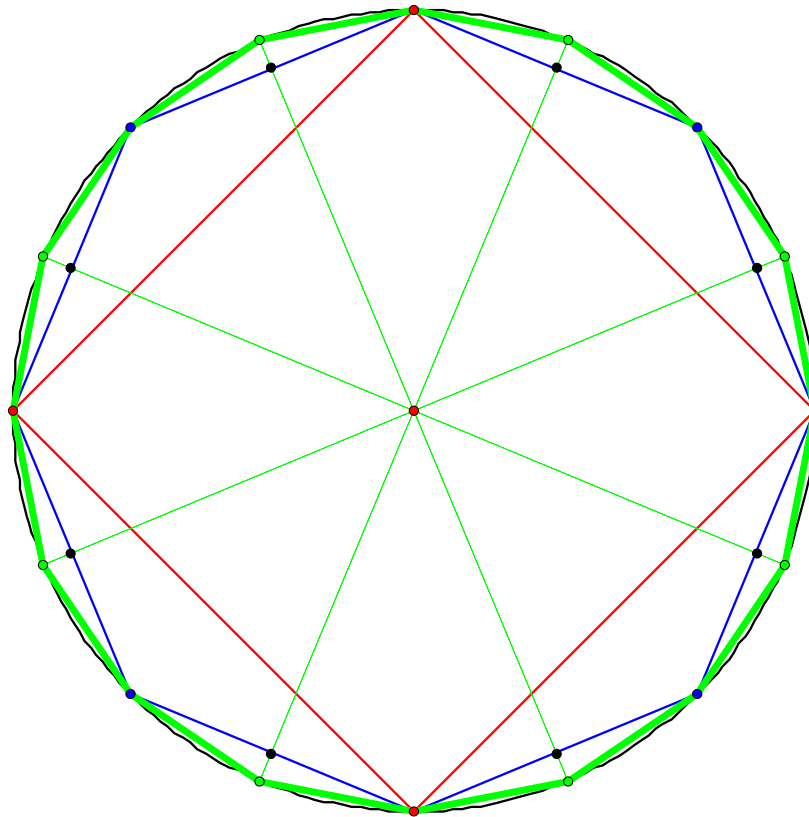
Connect the  
vertices.

- In general, a  $2^n$ -gon ( $n \geq 3$ ) can be inscribed in a circle by inscribing a  $2^{n-1}$ -gon, bisecting its sides, constructing radii through the points of bisection, and then connecting the vertices on the circle to form the  $2^n$ -gon.



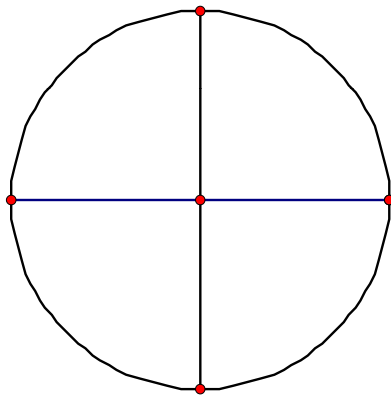
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Note that for large  $n$ , the area of the inscribed  $2^n$ -gon is close to, but still less than, the area of the circle.

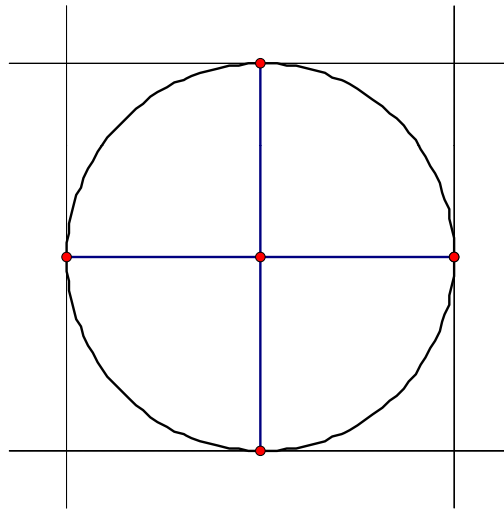


**Theorem:** A regular  $2^n$ -gon,  $n \geq 2$ , can be circumscribed about a circle.

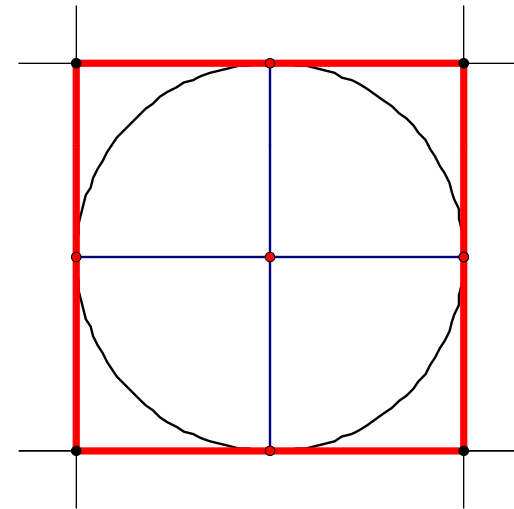
- Circumscribing a square about a circle:



Construct  $\perp$   
diagonals.

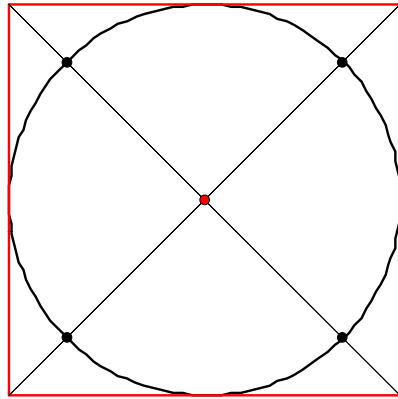


Construct  $\spadesuit$ 's  
at pts. of I  
with circle.

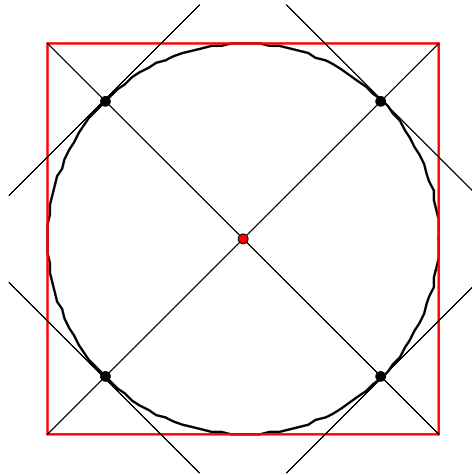


Construct pts. of  
I of  $\spadesuit$ 's & connect  
vertices.

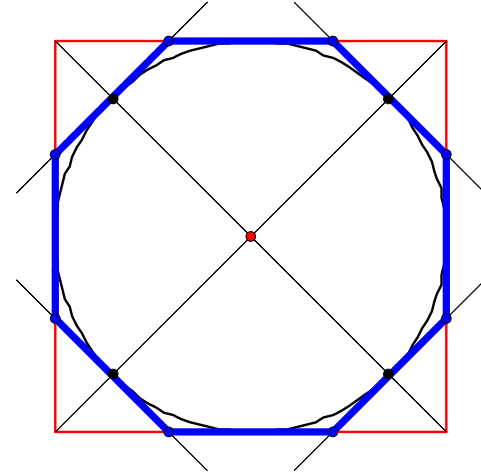
- Using the circumscribed square to circumscribe an octagon about a circle:



Construct pts.  
of I of circle  
& diagonals  
of the square.



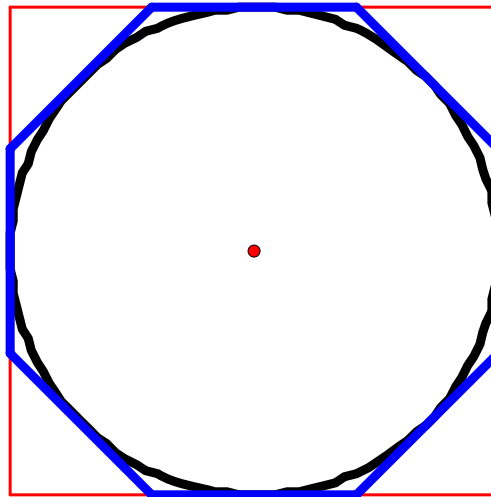
Construct ♠'s  
to diagonals  
at these pts.  
of I.

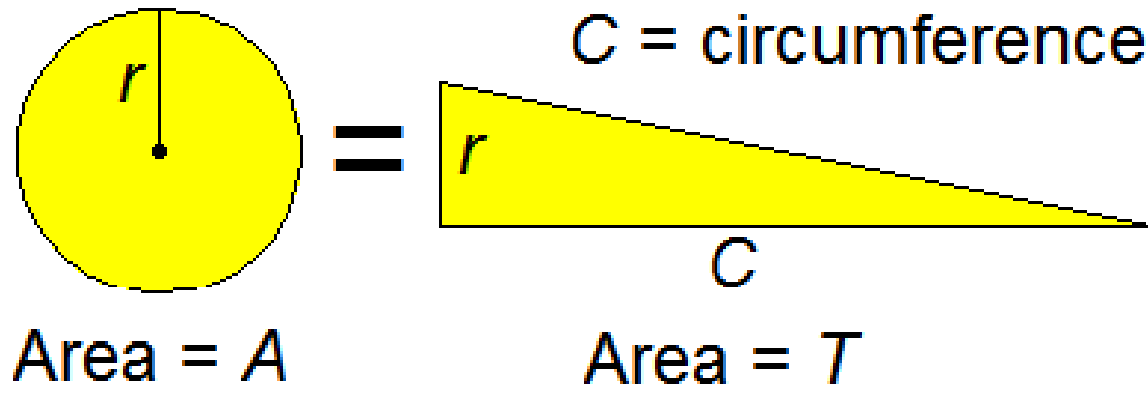


Connect pts.  
of I of ♠'s &  
the square.

- Continuing this process will circumscribe a  $2^n$ -gon ( $n \geq 3$ ) about a circle using a circumscribed  $2^{n-1}$ -gon.

Note that for large  $n$ , the area of the circumscribed  $2^n$ -gon is close to, but larger than, the area of the circle.





To prove that  $A = T$ , Archimedes had to rule out the cases:

**Case 1:**  $A > T$ .

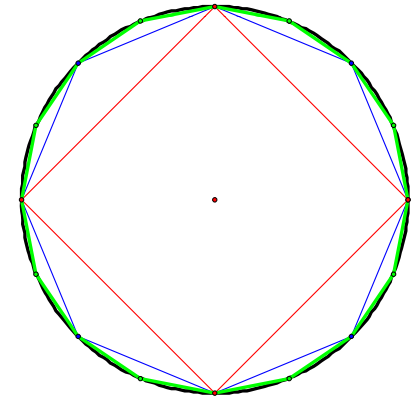
**Case 2:**  $A < T$ .

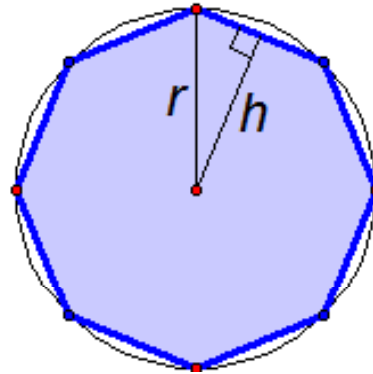
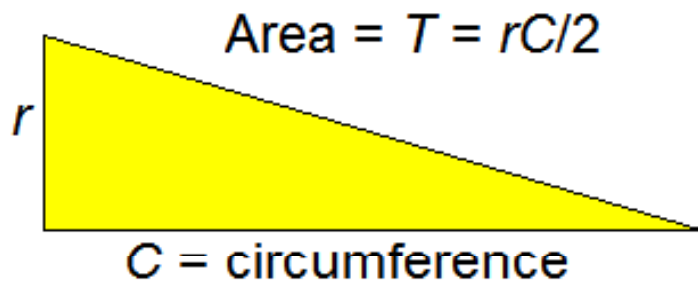
Then by the law of trichotomy,  $A$  must equal  $T$ .

# Case 1: $A > T$

By way of **contradiction**, assume that the area  $A$  of the circle is greater than the area  $T$  of the triangle.

Since we can inscribe  $2^n$ -gons in the circle whose areas approach the area of the circle, and since  $A > T$ , we will eventually reach a  $2^n$ -gon  $G$  whose area is between  $A$  and  $T$ . That is,  $A > G > T$ .





Area of an  $n$ -gon  $G$ :  
 Area =  $G = hQ/2$   
 where  $Q = \text{perimeter}$

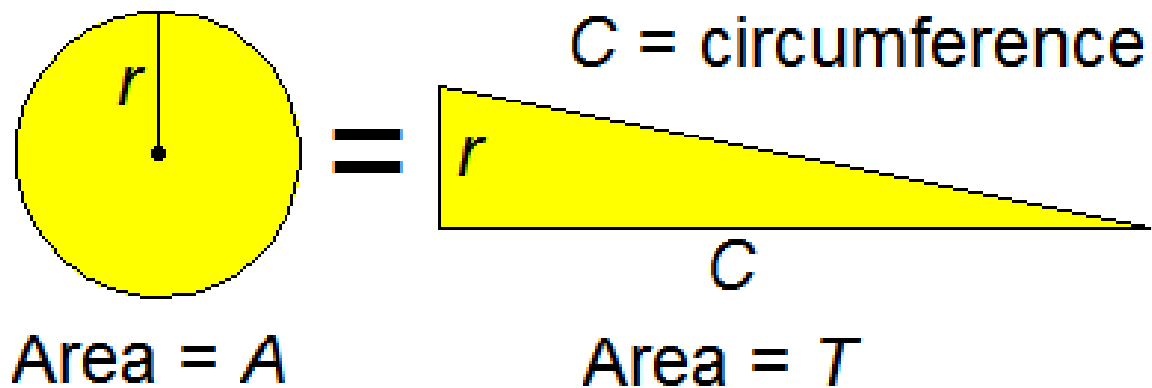
The area of the  $n$ -gon  $G$  is  $hQ/2$  where  $Q$  is the perimeter of the  $n$ -gon. Since  $G$  is inscribed in the circle, its perimeter  $Q$  is less than the circumference  $C$  of the circle. So  $G < hC/2$ . Also,  $h$  is less than  $r$ . Hence  $G < rC/2 = T$ . But this **contradicts** the “established fact” that  **$G > T$** .

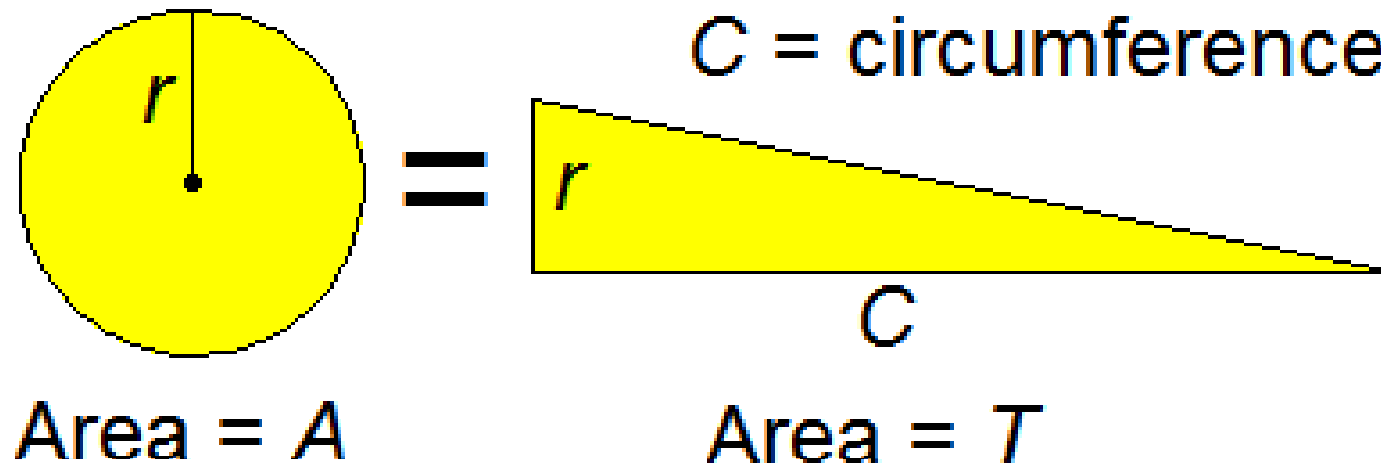
So the **assumption that  $A > T$**  cannot be correct. So we must conclude that  $A \leq T$ .

## Case 2: $A < T$

In a similar fashion, using circumscribed  $n$ -gons, it can be shown that this case is also impossible.

So by the trichotomy law, the only remaining possibility is that  $A = T$ .





Although Archimedes reduced the problem of finding the area of a circle to that of finding the known area of a triangle, he had no way of constructing this triangle.

In a previous lesson you saw that Ferdinand Lindemann (1852 – 1939) proved that  $\pi$  was not a constructible number. So Archimedes had no way of constructing the circumference ( $C = 2\pi r$ ) that formed the base of his triangle.

He solved this problem by estimating  $\pi$  as the ratio of the circumference of a circle to its diameter.

What he found is that:

The ratio of the circumference of a circle to its diameter is between  $3\frac{10}{71}$  and  $3\frac{1}{7}$ .

This is equivalent to saying that  $\pi$  can be approximated by

$$3.140845 < \pi < 3.142857.$$

So Archimedes successfully nailed down the approximation of  $\pi$  to the two decimal place accuracy of 3.14.