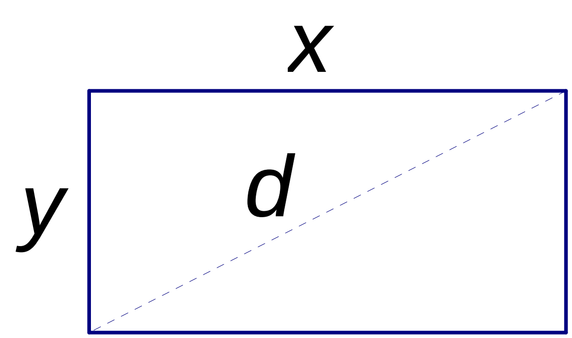
Standard Babylonian Problem:

You want to construct a rectangle having a specified area and a specified diagonal. What are the sides of this rectangle?

Given: $xy = a$

$$x^2 + y^2 = d^2$$

Find: x and y



Here's how the Babylonians solved the system of equations:

$$xy = a \quad \text{and} \quad x^2 + y^2 = d^2$$

They recognized that

$$(x+y)^2 = x^2 + 2xy + y^2 = d^2 + 2a$$

So $$x + y = \sqrt{d^2 + 2a}.$$

In a similar fashion, expanding $(x - y)^2$ yields $x - y = \sqrt{d^2 - 2a}$.

So the given system of equations

$$xy = a \text{ and } x^2 + y^2 = d^2$$

is equivalent to the system

$$x + y = \sqrt{d^2 + 2a}$$
$$x - y = \sqrt{d^2 - 2a}.$$

$$x + y = \sqrt{d^2 + 2a}$$
$$x - y = \sqrt{d^2 - 2a}.$$

Adding these equations and solving for *x* gives

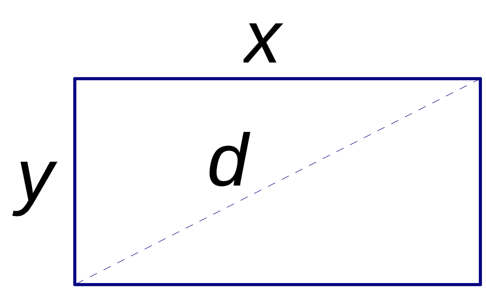
$$x = \frac{1}{2}\left(\sqrt{d^2 + 2a} + \sqrt{d^2 - 2a}\right),$$

and subtracting them yields

$$y = \frac{1}{2}\left(\sqrt{d^2 + 2a} - \sqrt{d^2 - 2a}\right).$$

Conclusion:

A rectangle having area *a* and diagonal *d* has sides



$$x = \frac{1}{2}\left(\sqrt{d^2 + 2a} + \sqrt{d^2 - 2a}\right)$$

$$y = \frac{1}{2}\left(\sqrt{d^2 + 2a} - \sqrt{d^2 - 2a}\right).$$

Example:

What are the measures of the sides of a rectangle having area $a = 12$ and whose diagonal measures $d = 5$?

$$x = \frac{1}{2}\left(\sqrt{d^2 + 2a} + \sqrt{d^2 - 2a}\right) \qquad y = \frac{1}{2}\left(\sqrt{d^2 + 2a} - \sqrt{d^2 - 2a}\right)$$

$$x = \frac{1}{2}\left(\sqrt{5^2 + 2 \cdot 12} + \sqrt{5^2 - 2 \cdot 12}\right) \qquad y = \frac{1}{2}\left(\sqrt{5^2 + 2 \cdot 12} - \sqrt{5^2 - 2 \cdot 12}\right)$$

$$= \frac{1}{2}\left(\sqrt{49} + \sqrt{1}\right) = 4 \qquad = \frac{1}{2}\left(\sqrt{49} - \sqrt{1}\right) = 3$$

Another example:

What are the measures of the sides of a rectangle having area $a = 12$ and whose diagonal measures $d = 6$?

$$x = \frac{1}{2}\left(\sqrt{d^2 + 2a} + \sqrt{d^2 - 2a}\right) \qquad y = \frac{1}{2}\left(\sqrt{d^2 + 2a} - \sqrt{d^2 - 2a}\right)$$

$$x = \frac{1}{2}\left(\sqrt{6^2 + 2\cdot 12} + \sqrt{6^2 - 2\cdot 12}\right) \qquad y = \frac{1}{2}\left(\sqrt{6^2 + 2\cdot 12} - \sqrt{6^2 - 2\cdot 12}\right)$$

$$= \frac{1}{2}\left(\sqrt{60} + \sqrt{12}\right) = \ ? \qquad = \frac{1}{2}\left(\sqrt{60} - \sqrt{12}\right) = \ ?$$

To approximate $\sqrt{n}$ the Babylonian way:

- First find the largest perfect square a^2 that is less than n.
- Then expressed n in the form $n = a^2 + b$.
- And then used the "formula"

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}$$

Examples of using this formula:

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}$$

$49 = 7^2$ is the largest perfect square less than 60, so

$$\sqrt{60} = \sqrt{7^2 + 11} \approx 7 + \frac{11}{2 \cdot 7} = 7\frac{11}{14}$$

Similarly,

$$\sqrt{12} = \sqrt{3^2 + 3} \approx 3 + \frac{3}{2 \cdot 3} = 3\frac{1}{2}$$

Conclusion

What are the measures of the sides of a rectangle having area $a = 12$ and whose diagonal measures $d = 6$?

$$x = \frac{1}{2}\left(\sqrt{6^2 + 2\cdot 12} + \sqrt{6^2 - 2\cdot 12}\right)$$
$$= \frac{1}{2}\left(\sqrt{60} + \sqrt{12}\right)$$
$$\approx \frac{1}{2}\left(7\frac{11}{14} + 3\frac{1}{2}\right) = 5\frac{9}{14}$$

$$y = \frac{1}{2}\left(\sqrt{6^2 + 2\cdot 12} - \sqrt{6^2 - 2\cdot 12}\right)$$
$$= \frac{1}{2}\left(\sqrt{60} - \sqrt{12}\right)$$
$$\approx \frac{1}{2}\left(7\frac{11}{14} - 3\frac{1}{2}\right) = 2\frac{1}{7}$$

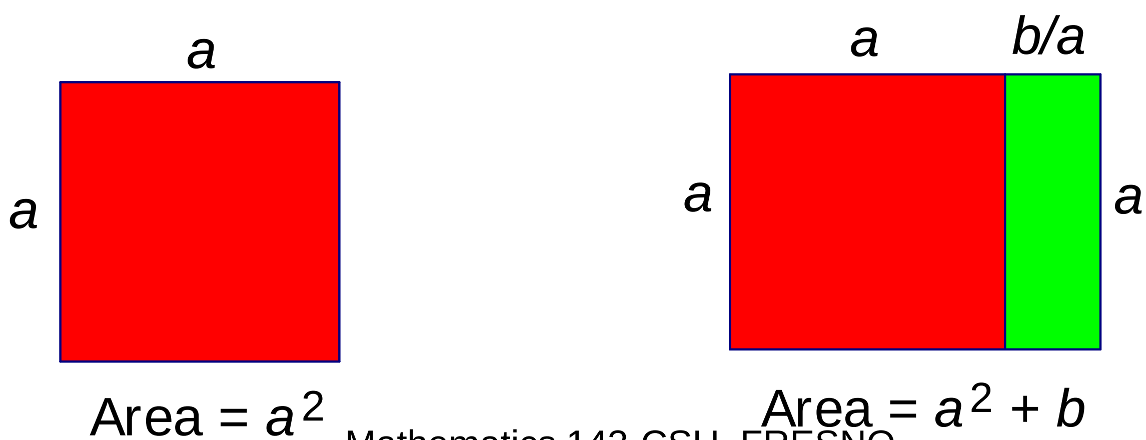
Babylonian proof that

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}$$

Start with a square having side *a*.

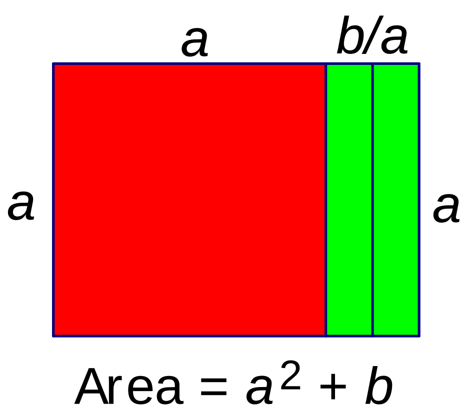
Attach a rectangle measuring *b*/*a* by *a.*

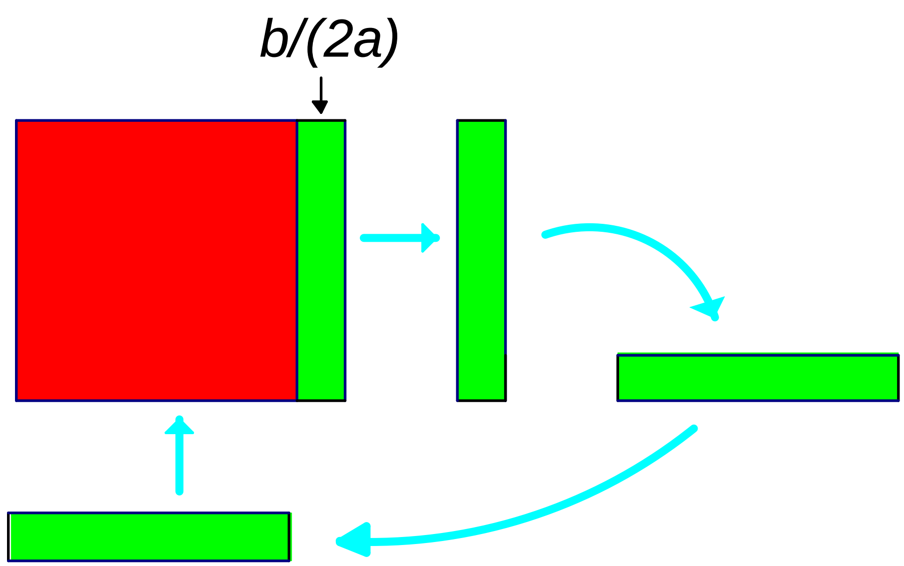
The area of the resulting figure is $a^2 + b$.



Divide the smaller attached rectangle into two rectangles each measuring *b*/2*a* by *a*.

Remove one of the *b*/2*a* by *a* rectangles, rotate it, and attach it to the bottom of the square having side *a*.

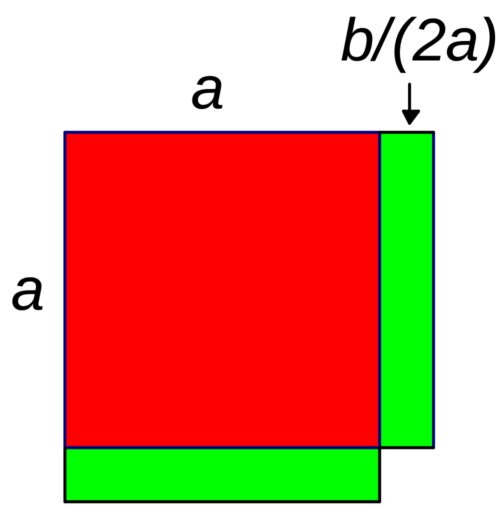




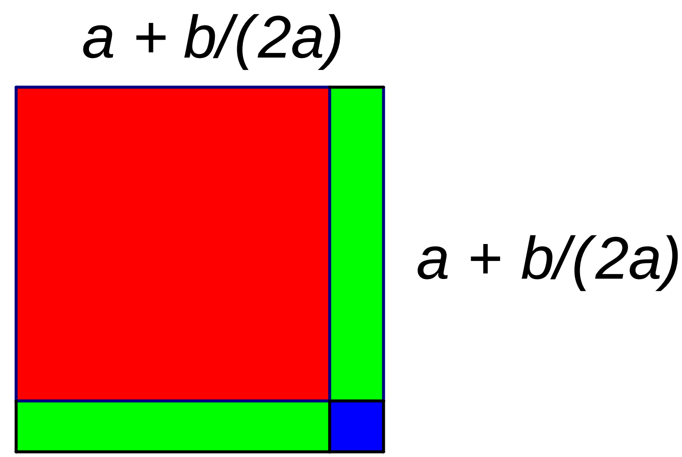
The area of the resulting figure is still $a^2 + b$.

Enclose this figure in a square.

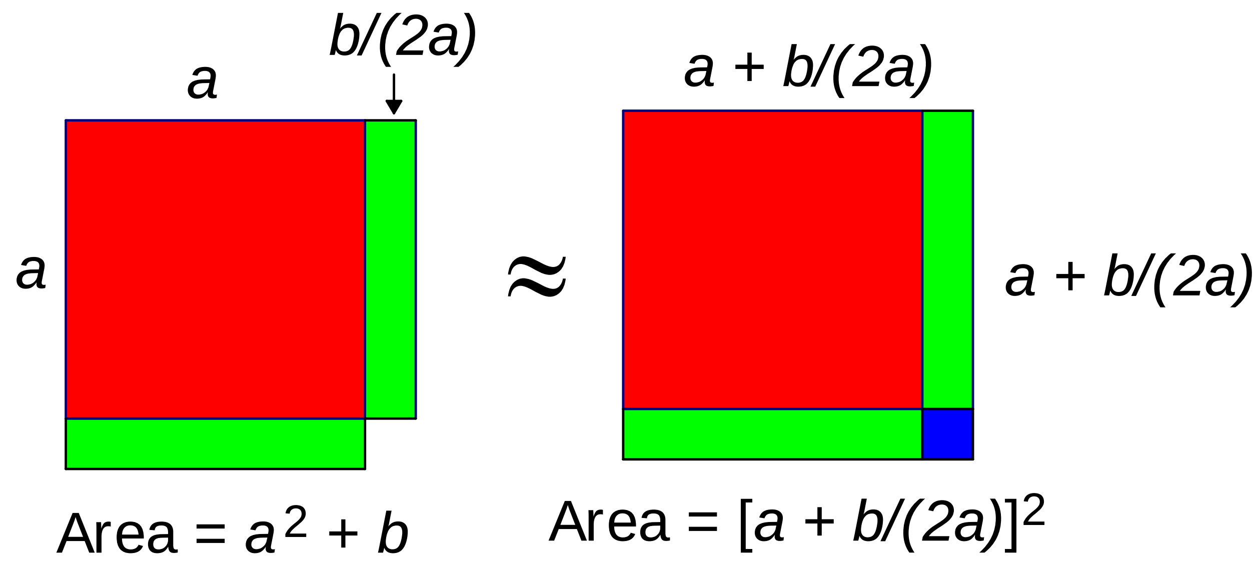
The sides of this square measure $a + b/2$a.



Area = $a^2 + b$



Area = $[a + b/(2a)]^2$



The areas of these two figures are approximately equal, so

$$\sqrt{a^2+b} \approx \sqrt{\left(a+b/2a\right)^2} = a + \frac{b}{2a}$$