What's a depressed polynomial equation?

An n^{th} degree polynomial equation is said to be **depressed** if it is missing the $(n - 1)^{st}$ term. For example:

$$x^2 - 9 = 0$$

$$x^3 + 8x = 9$$

$$x^4 - 10x^2 + 4x + 8 = 0$$

A depressed quadratic equation is quite simple to solve.

$$x^2 - c = 0 \quad \Rightarrow \quad x = \pm\sqrt{c}$$

And as you will see in later, there are techniques for solving depressed cubic and quartic equations.

Depressing an Equation

Substituting $x = y - (b/na)$ in the equation

$$ax^n + bx^{n-1} + \cdots + c = 0$$

will result in a n^{th} degree, depressed equation in the variable y.

Once the depressed equation is solved, the substitution $x = y - (b/na)$ can then be used to solve for x.

Here's what the substitution $x = y - (b/2a)$ does to a quadratic equation.

$$ax^2 + bx + c = 0 \;\Rightarrow$$

$$a(y - b/2a)^2 + b(y - b/2a) + c = 0 \;\Rightarrow$$

$$\vdots$$

$$ay^2 + \frac{4ac - b^2}{4a} = 0 \;\Rightarrow$$

$$y^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \quad y = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Since we substituted $x = y - b/2a$, the solution to the quadratic equation $ax^2 + bx + c = 0$ is

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solve $x^3 + 6x^2 + 3x = 10$

Making the substitution $x = y - 6/3\cdot 1$,

$$(y-2)^3 + 6(y-2)^2 + 3(y-2) = 10 \Rightarrow$$

$$\vdots$$

$$y^3 - 9y = 0 \Rightarrow$$

$$y(y^2 - 9) = 0 \Rightarrow$$

$$y = 0, 3, -3 \Rightarrow$$

$$x = y - 2 = -2, 1, -5$$

Solve the quartic

$$x^4 + 12x^3 + 49x^2 + 70x + 40 = 0$$

Making the substitution $x = y - 12/4\cdot 1$,

$$(y-3)^4 + 12(y-3)^3 + 49(y-3)^2 + 70(y-3) + 40 = 0 \Rightarrow$$

$$\vdots$$

$$y^4 - 5y^2 + 4 = 0 \Rightarrow$$

$$(y^2-1)(y^2-4) = 0 \Rightarrow y = -1, 1, -2, 2$$

$$x = y - 3 = -4, -2, -5, -1$$

Not all cubic and quartic equations can be solved by solving the depressed equation as we did in the last two examples. It’s usually the case that the depressed equation can’t be solved using the techniques you learned in high school.

In the next lesson you will see how to solve any depressed cubic equation.

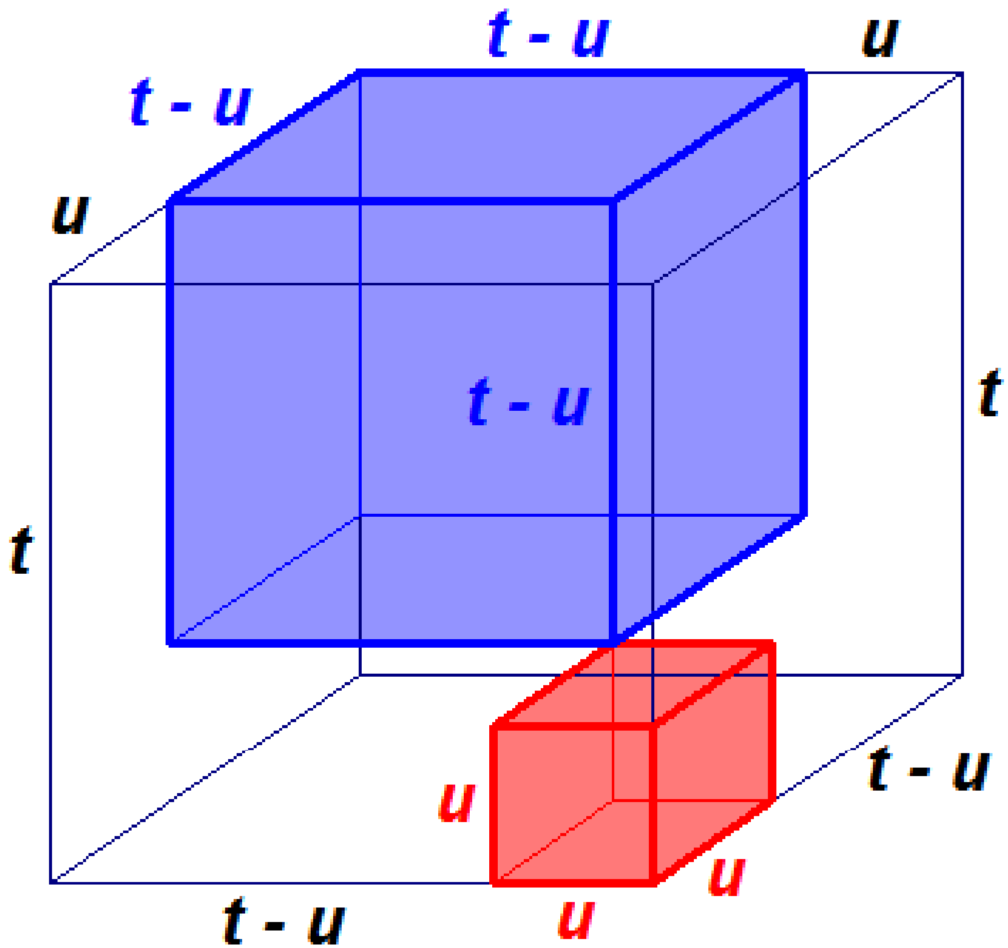
Now you've seen how to use the substitution $x = y - b/3a$ to convert the cubic equation $ax^3 + bx^2 + cx + d = 0$ into a depressed cubic equation $y^3 + my = n$.

And in the special case where $n = 0$, you could solve the depressed equation by simply factoring.

Now you will see how to solve the depressed cubic $y^3 + my = n$, independent of the values of m and n.

Actually, what we will do is derive Cardano's formula for finding one solution to the depressed cubic equation.

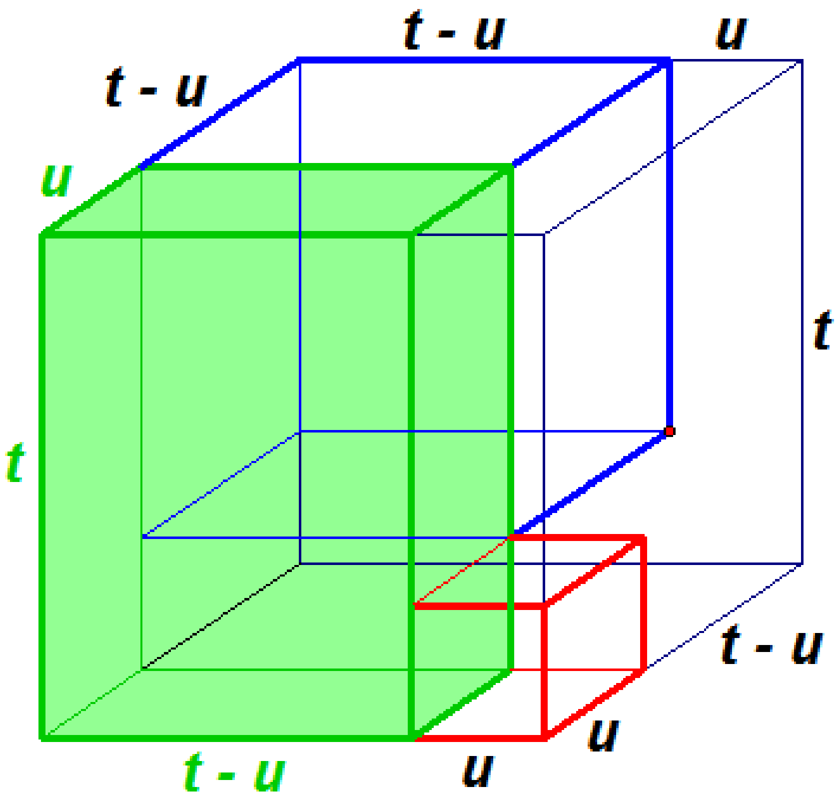
When Cardano wrote his proof in the 16th, he started by imagining a large cube having sides measuring t. Each side was divided into segments measuring $t - u$ and u in such a way that cubes could be constructed in diagonally opposite corners of the cube.



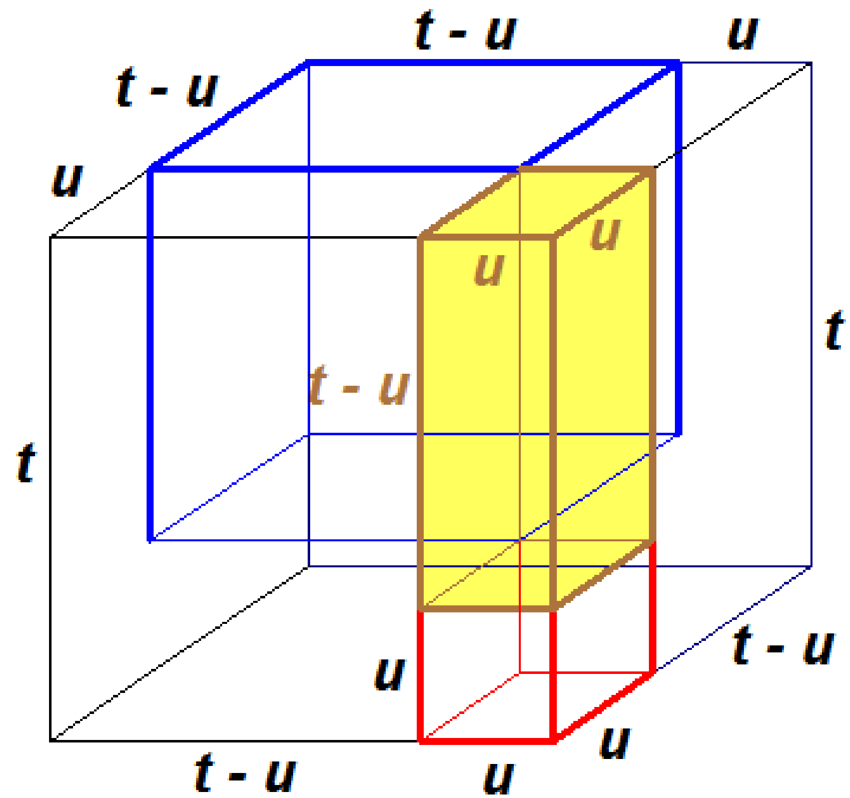
$$V_1 = (t - u)^3$$

$$V_2 = u^3$$

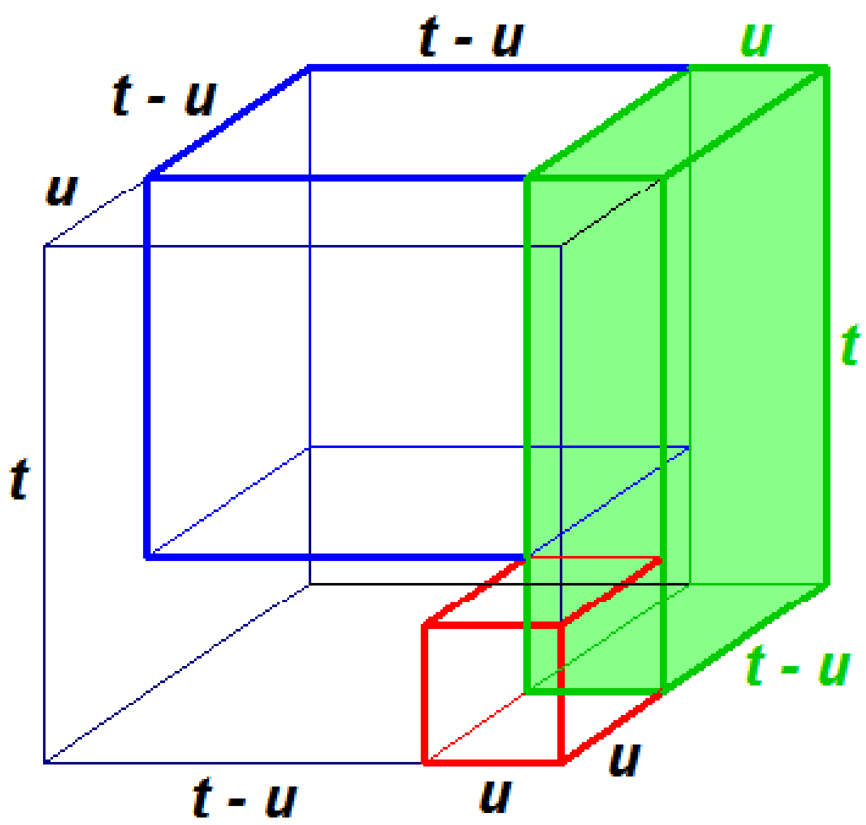
This divides the large cube into 6 parts, two of which are pictured here.



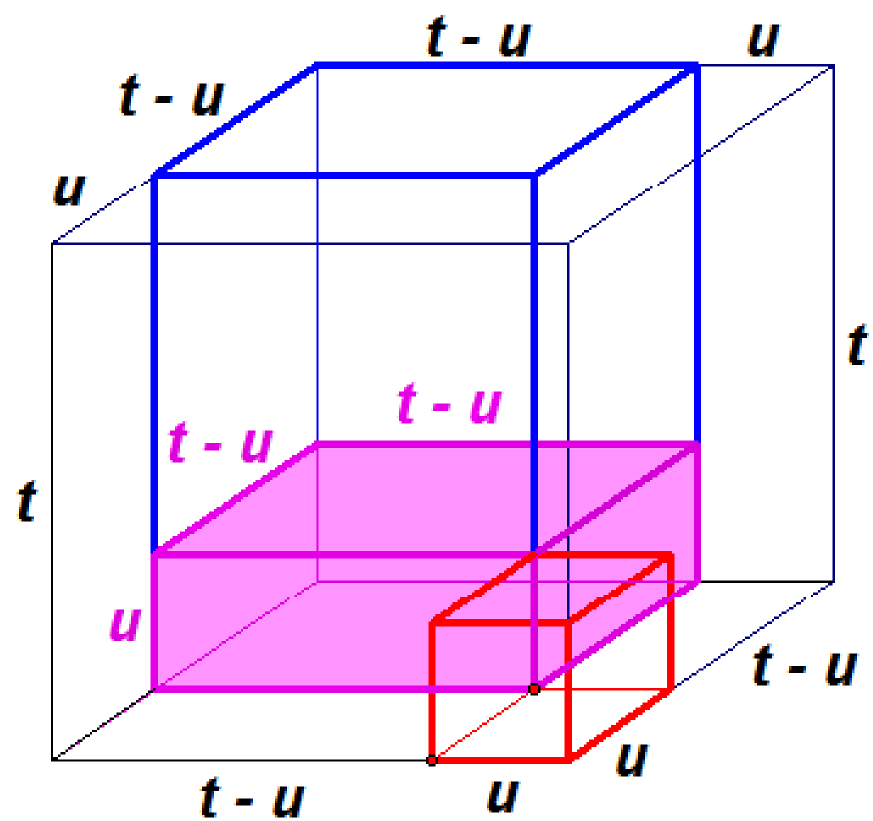
$V_3 = tu(t - u)$



$V_4 = (t - u)u^2$



$$V_5 = tu(t - u)$$



$$V_6 = u(t - u)^2$$

Since the volume t^3 of the large cube is equal to the sum of the volumes of its six parts, we get,

$$V_1 = (t - u)^3$$
$$V_2 = u^3$$
$$V_3 = tu(t - u)$$
$$V_4 = (t - u)u^2$$
$$V_5 = tu(t - u)$$
$$V_6 = u(t - u)^2$$

$$t^3 = (t-u)^3 + u^3 + 2tu(t-u) + (t-u)u^2 + u(t-u)^2$$

which can be expressed as

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3.$$

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3$$

This is reminiscent of the depressed cubic $y^3 + my = n$ we want to solve. So set

$$y = t - u, \quad m = 3tu, \text{ and } \quad n = t^3 - u^3.$$

Substituting $u = m/3t$ into $n = t^3 - u^3$,

gives $t^3 - \dfrac{m^3}{27t^3} = n$ which simplifies to

$$t^6 - nt^3 - \frac{m^3}{27} = 0.$$

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

$y = t - u,$
$m = 3tu,$
$n = t^3 - u^3$

But this is a quadratic in t^3. So using only the positive square root we get,

$$t^3 = \frac{n + \sqrt{n^2 + \frac{4m^3}{27}}}{2} = \cdots = \frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} \Rightarrow$$

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

$$t^3 = \frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}$$

$y = t - u,$

$m = 3tu,$

$n = t^3 - u^3$

And since $u^3 = t^3 - n$, we get

$$u^3 = \frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} - n \quad \text{or}$$

$$u = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} \qquad u = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

$$y = t - u, \quad m = 3tu, \quad n = t^3 - u^3$$

Since $y = t - u$, we now have Cardano's formula for solving the depressed cubic.

$$y^3 + my = n \quad \Rightarrow$$

$$y = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

Example: Find all solutions to

$$x^3 - 9x^2 + 24x - 20 = 0$$

- Substitute $x = y - b/3a$ to depress the equation $ax^3 + bx^2 + cx + d = 0$.

$$x^3 - 9x^2 + 24x - 20 = 0$$

$$\vdots \quad \text{sub. } x = y + 3$$

$$y^3 - 3y = 2$$

- Use Cardano's formula

$$y^3 + my = n \quad \Rightarrow \quad y = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

to solve the depressed equation.

$$y^3 - 3y = 2 \qquad m = -3,\ n = 2$$

$$\sqrt[3]{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} = \sqrt[3]{1-1} = 0$$

$$y = \sqrt[3]{1} - \sqrt[3]{-1} = 1 - (-1) = 2$$

- Use algebra to find, if possible, the other solutions to the depressed equation.

$y = 2$ is a solution to $y^3 - 3y = 2$, so
$(y - 2)$ is a factor of $y^3 - 3y - 2$.

$$
\begin{array}{r}
y^2 + 2y + 1 \\
y - 2 \,\overline{\smash{\big)}\, y^3 + 0y^2 - 3y - 2} \\
\underline{y^3 - 2y^2} \phantom{{} - 3y - 2} \\
2y^2 - 3y \phantom{{} - 2} \\
\underline{2y^2 - 4y} \phantom{{} - 2} \\
y - 2 \\
\underline{y - 2} \\
0
\end{array}
$$

$$y^3 - 3y = 2 \quad \Rightarrow \quad (y-2)(y^2+2y+1) = 0$$

$$\Rightarrow \quad (y-2)(y+1)^2 = 0$$

$$\Rightarrow \quad y = 2,\ -1$$

- Use the substitution $y = b/3a$ to find the solutions to the original equation.

$$y = 2,\ -1 \quad \Rightarrow \quad x = y + 3 = 5,\ 2$$