Decartes discovered a rule which helps determine the number of positive and negative real roots of a polynomial function having real coefficients.

To use this rule, the polynomial function must be written in descending or ascending order and the constant term must be nonzero. Examples:

$$f(x) = 5x^4 - 3x^3 + 7x^2 - 12x + 4$$

$$g(x) = 6 + 4x - 21x^2 + x^4$$

Decartes' rule of signs cannot be used with the polynomial function

$$p(x) = 6x^6 - 5x - 2x^2$$

since the terms are not in order and since the constant term is zero.

But $p(x) = x \cdot h(x)$ where

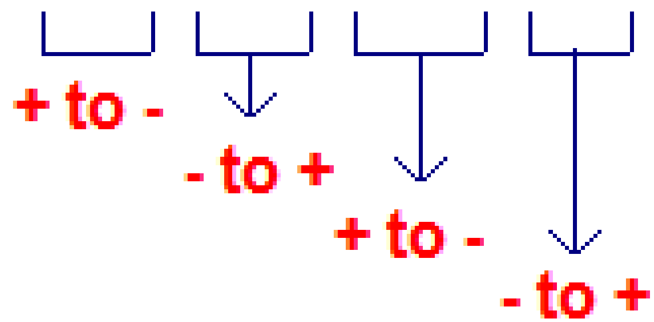
$$h(x) = 6x^5 - 2x - 5$$

is in the proper form for using Decartes' rule.

Variations of sign

When a polynomial function is written in the form for using Decartes' rule, the ***variations of sign*** of the function is the number of times the sign changes.

$$f(x) = 5x^4 - 3x^3 + 7x^2 - 12x + 4$$



4 variations

$$g(x) = 6 + 4x - 21x^2 + x^4$$

+ to +

+ to -

- to +

2 variations

$$h(x) = 6x^5 - 2x - 5$$

+ to -

- to -

1 variation

$$k(x) = x^4 + 2x^2 + 3$$

+ to +

+ to +

no variations

Decartes’ Rule of Signs

The number of positive real roots of a polynomial function $P(x)$ with real coefficients and a nonzero constant term is either equal to

- the number of variations of sign of $P(x)$

or

- the number of variations of sign minus an even number.

Example

Find the possible number of positive roots of $f(x) = 5x^4 - 3x^3 + 7x^2 - 12x + 4.$

There are 4 variations of sign. So by Decartes' rule, the number of positive roots is 4 or 4 – 2 or 4 – 4. That is, $f(x)$ has 4, 2, or 0 positive roots.

Using Decartes' rule to find the number of negative roots

Since the negative roots of $P(x)$ are the positive roots of $P(-x)$, the negative roots of $P(x)$ can be found by applying Decartes' rule to $P(-x)$.

Example

Find the possible number of negative roots of $f(x) = 5x^4 - 3x^3 + 7x^2 - 12x + 4$.

$f(-x) = 5x^4 + 3x^3 + 7x^2 + 12x + 4$ has no variations of sign. So by Decartes' rule, $f(-x)$ has no positive roots. Thus $f(x)$ has no negative roots.

Since $f(x) = 5x^4 - 3x^3 + 7x^2 - 12x + 4$ has 4, 2, or 0 positive roots and no negative roots, the equation

$$5x^4 - 3x^3 + 7x^2 - 12x + 4 = 0$$

has exactly 4 positive real solutions, or 2 positive real solutions and two complex solutions, of exactly 4 complex solutions.

Example: Describe the solutions to

$$6x^5 - 2x - 5 = 0.$$

$h(x) = 6x^5 - 2x - 5$	1 variation
$h(-x) = -6x^5 + 2x - 5$	2 variations

	pos. real	neg. real	complex
	1	2	2
or	1	0	4

Example: Describe the solutions to

$$x^4 + 2x^2 + 3 = 0$$

$$k(x) = x^4 + 2x^2 + 3$$ no variations

$$k(-x) = x^4 + 2x^2 + 3$$ no variations

By Decartes' rule, *k*(*x*) has no positive and no negative real roots. Since 0 is clearly not a solution to the equation, all four solutions must be complex numbers.