THE CHINESE TEACHERS' APPROACH TO THE MEANING OF DIVISION BY FRACTIONS

The deficiency in the subject matter knowledge of the U.S. teachers on the advanced arithmetical topic of division by fractions did not appear among the Chinese teachers. While only one among the 23 U.S. teachers generated a conceptually correct representation for the meaning of the equation, 90% of the Chinese teachers did. Sixty-five of the 72 Chinese teachers created a total of more than 80 story problems representing the meaning of division by a fraction. Twelve teachers proposed more than one story to approach different aspects of the meaning of the operation. Only six (8%) teachers said that they were not able to create a story problem, and one teacher provided an incorrect story (which represented $\frac{1}{2} \div 1\frac{3}{4}$ rather than $1\frac{3}{4} \div \frac{1}{2}$). Figure 3.1 displays a comparison of teachers' knowledge about this topic.

The Chinese teachers represented the concept using three different models of division: measurement (or quotitive), partitive, and product and factors.[6] For example, $1\frac{3}{4} \div \frac{1}{2}$ might represent:

- $1\frac{3}{4}$ feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ (measurement model)
- $1\frac{3}{4}$ feet $\div \frac{1}{2} = \frac{7}{2}$ feet (partitive model)
- $1\frac{3}{4}$ square feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ feet (product and factors)

which might correspond to:

- How many $\frac{1}{2}$-foot lengths are there in something that is 1 and $\frac{3}{4}$ feet long?
- If half a length is 1 and $\frac{3}{4}$ feet, how long is the whole?
- If one side of a $1\frac{3}{4}$ square foot rectangle is $\frac{1}{2}$ feet, how long is the other side?

The Models of Division by Fractions

The Measurement Model of Division: "Finding How Many $\frac{1}{2}$s There Are in $1\frac{3}{4}$" or "Finding How Many Times $1\frac{3}{4}$ is of $\frac{1}{2}$"

Sixteen stories generated by the teachers illustrated two ideas related to the measurement model of division: "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." Eight stories about five topics corresponded to "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$." Here are two examples:

[6]Greer (1992) gives an extensive discussion of models of multiplication and division. His category "rectangular area" is included in "product and factors."

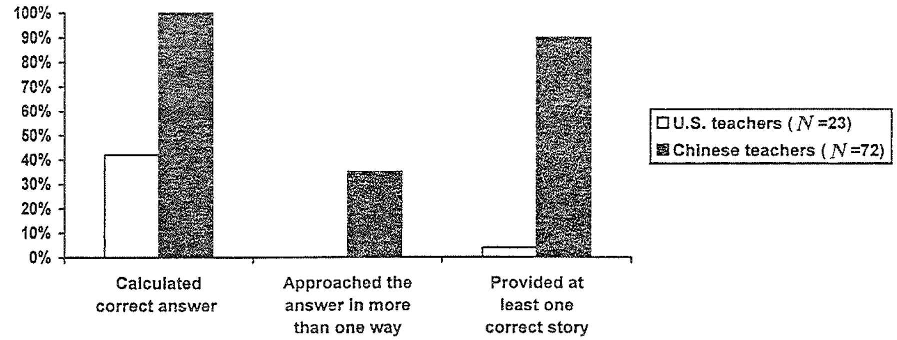

FIG. 3.1. Teachers' knowledge of division by fractions.

> Illustrating it with the measurement model of division, $1\frac{3}{4} \div \frac{1}{2}$ can be articulated as how many $\frac{1}{2}$s there are in $1\frac{3}{4}$. To represent it we can say, for example, given that a team of workers construct $\frac{1}{2}$ km of road each day, how many days will it take them to construct a road of $1\frac{3}{4}$ km long? The problem here is to find how many pieces of $\frac{1}{2}$ km, which they can accomplish each day, are contained in $1\frac{3}{4}$ km. You divide $1\frac{3}{4}$ by $\frac{1}{2}$ and the result is $3\frac{1}{2}$ days. It will take them $3\frac{1}{2}$ days to construct the road. (Tr. R.)

> Cut an apple into four pieces evenly. Get three pieces and put them together with a whole apple. Given that $\frac{1}{2}$ apple will be a serving, how many servings can we get from the $1\frac{3}{4}$ apples? (Ms. I.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" parallels the approach of Tr. Belle, the U.S. teacher who had a conceptual understanding of the topic. There were eight other stories that represented "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." For example:

> It was planned to spend $1\frac{3}{4}$ months to construct a bridge. But actually it only took $\frac{1}{2}$ month. How many times is the time that was planned of the time that actually was taken? (Tr. K.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$" are two approaches to the measurement model of division by fractions. Tr. Li indicated that though the measurement model is consistent for whole numbers and fractions when fractions are introduced the model needs to be revised:

> In whole number division we have a model of finding how many times one number is of another number. For example, how many times the number 10 is of the number 2? We divide 10 by 2 and get 5. 10 is 5 times 2. This is what we call the measurement model. With fractions, we can still say, for example, what times $\frac{1}{2}$ is $1\frac{3}{4}$? Making a story problem, we can say for instance, there are two fields. Field A is $1\frac{3}{4}$ hectares, and field B is $\frac{1}{2}$ hectare. What

> times the area of field B is the area of field A? To calculate the problem we divide $1\frac{3}{4}$ hectares by $\frac{1}{2}$ hectare and get $3\frac{1}{2}$. Then we know that the area of the field A is $3\frac{1}{2}$ times that of the field B. The equation you asked me to represent fits this model. However, when fractions are used this division model of measurement need to be revised. In particular, when the dividend is smaller than the divisor and then the quotient becomes a proper fraction. Then the model should be revised. The statement of "finding what fraction one number is of another number," or, "finding what fractional times one number is of another number" should be added on the original statement. For example, for the expression $2 \div 10$, we may ask, what fraction of 10 is 2? Or, what fractional times is 2 of 10? We divide 2 by 10 and get $\frac{1}{5}$. 2 is $\frac{1}{5}$ of 10. Similarly, we can also ask: What is the fractional part that $\frac{1}{4}$ is of $1\frac{1}{2}$? Then you should divide $\frac{1}{4}$ by $1\frac{1}{2}$ and get $\frac{1}{6}$.

The Partitive Model of Division: Finding a Number Such That $\frac{1}{2}$ of It is $1\frac{3}{4}$

Among more than 80 story problems representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$, 62 stories represented the partitive model of division by fractions—"finding a number such that $\frac{1}{2}$ of it is $1\frac{3}{4}$":

> Division is the inverse of multiplication. Multiplying by a fraction means that we know a number that represents a whole and want to find a number that represents a certain fraction of that. For example, given that we want to know what number represents $\frac{1}{2}$ of $1\frac{3}{4}$, we multiply $1\frac{3}{4}$ by $\frac{1}{2}$ and get $\frac{7}{8}$. In other words, the whole is $1\frac{3}{4}$, and $\frac{1}{2}$ of it is $\frac{7}{8}$. In division by a fraction, on the other hand, the number that represents the whole becomes the unknown to be found. We know a fractional part of it and want to find the number that represents the whole. For example, $\frac{1}{2}$ of a jump-rope is $1\frac{3}{4}$ meters, what is the length of the whole rope? We know that a part of a rope is $1\frac{3}{4}$ meters, and we also know that this part is $\frac{1}{2}$ of the rope. We divide the number of the part, $1\frac{3}{4}$ meters, by the corresponding fraction of the whole, $\frac{1}{2}$, we get the number representing the whole, $3\frac{1}{2}$ meters. Dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we will find that the whole rope is $3\frac{1}{2}$ meters long . . . But I prefer not to use dividing by $\frac{1}{2}$ to illustrate the meaning of division by fractions. Because one can easily see the answer without really doing division by fractions. If we say $\frac{4}{5}$ of a jump-rope is $1\frac{3}{4}$ meters, how long is the whole rope? The division operation will be more significant because then you can't see the answer immediately. The best way to calculate it is to divide $1\frac{3}{4}$ by $\frac{4}{5}$ and get $2\frac{3}{16}$ meters. (Ms. G.)

> Dividing by a fraction is finding a number when a fractional part of it is known. For example, given that we know that $\frac{1}{2}$ of a number is $1\frac{3}{4}$, dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we can find out that this number is $3\frac{1}{2}$. Making a story problem to illustrate this model, let's say that one kind of wood weighs $1\frac{3}{4}$ tons per cubic meter, it is just $\frac{1}{2}$ of the weight of per cubic meter of one kind of marble. How much does one cubic meter of the marble weigh? So we know that $\frac{1}{2}$ cubic meter of the marble weighs $1\frac{3}{4}$ tons. To find the weight of one cubic meter of it, we divide $1\frac{3}{4}$, the number that represents the fractional part, by $\frac{1}{2}$, the fraction which $1\frac{3}{4}$ represents, and get $3\frac{1}{2}$, the number of the whole. Per cubic meter the marble weighs $3\frac{1}{2}$ tons. (Tr. D.)

> My story will be: A train goes back and forth between two stations. From Station A to Station B is uphill and from Station B back to Station A is downhill. The train takes $1\frac{3}{4}$ hours going from Station B to Station A. It is only $\frac{1}{2}$ time of that from Station A to Station B. How long does the train take going from Station A to Station B? (Tr. S.)

> The mom bought a box of candy. She gave $\frac{1}{2}$ of it which weighed $1\frac{3}{4}$ kg to the grandma. How much did the box of the candy originally weigh? (Ms. M.)

The teachers above explained the fractional version of the partitive model of division. Tr. Mao discussed in particular how the partitive model of division by integers is revised when fractions are introduced:

> With integers students have learned the partitive model of division. It is a model of finding the size of each of the equal groups that have been formed from a given quantity. For example, in our class we have 48 students, they have been formed into 4 groups of equal size, how many students are there in each group? Here we know the quantity of several groups, 48 students. We also know the number of groups, 4. What to be found is the size of one group. So, *a partitive model is finding the value of a unit when the value of several units is known.* In division by fractions, however, the condition has been changed. Now what is known is not the value of several units, rather, the value of a part of the unit. For example, given that we paid $1\frac{3}{4}$ Yuan to buy $\frac{1}{2}$ of a cake, how much would a whole cake cost? Since we know that $\frac{1}{2}$ of the whole price is $1\frac{3}{4}$ Yuan, to know the whole price we divide $1\frac{3}{4}$ by $\frac{1}{2}$ and get $3\frac{1}{2}$ Yuan. In other words, *the fractional version of the partitive model is to find a number when a part of it is known.* (italics added)

Tr. Mao's observation was true. Finding a number when several units is known and finding a number when a fractional part of it is known are represented by a common model—finding the number that represents a unit when a certain amount of the unit is known. What differs is the feature of the amount: with a whole number divisor, the condition is that "*several times* the unit is known," but with a fractional divisor the condition is that "*a fraction* of the unit is known." Therefore, conceptually, these two approaches are identical.

This change in meaning is particular to the partitive model. In the measurement model and the factors and product model, division by fractions keeps the same meaning as whole number division. This may explain why so many of the Chinese teachers' representations were partitive.

Factors and Product: Finding a Factor That Multiplied by $\frac{1}{2}$ Will Make $1\frac{3}{4}$

Three teachers described a more general model of division—to find a factor when the product and another factor are known. The teachers articulated it as "to find a factor that when multiplied by $\frac{1}{2}$ makes $1\frac{3}{4}$":

> As the inverse operation of multiplication, division is to find the number representing a factor when the product and the other factor are known. From this perspective, we can get a word problem like "Given that the product of $\frac{1}{2}$ and another factor is $1\frac{3}{4}$, what is the other factor?" (Tr. M.)

> We know that the area of a rectangle is the product of length and width. Let's say that the area of a rectangle board is $1\frac{3}{4}$ square meters, its width is $\frac{1}{2}$ meters, what is its length? (Mr. A.)

These teachers regarded the relationship between multiplication and division in a more abstract way. They ignored the particular meaning of the multiplicand and multiplier in multiplication and related models of division. Rather, they perceived the multiplicand and multiplier as two factors with the same status. Their perspective, indeed, was legitimized by the commutative property of multiplication.

The concept of fractions as well as the operations with fractions taught in China and U.S. seem different. U.S. teachers tend to deal with "real" and "concrete" wholes (usually circular or rectangular shapes) and their fractions. Although Chinese teachers also use these shapes when they introduce the concept of a fraction, when they teach operations with fractions they tend to use "abstract" and "invisible" wholes (e.g., the length of a particular stretch of road, the length of time it takes to complete a task, the number of pages in a book).

Meaning of Multiplication by a Fraction: The Important Piece in the Knowledge Package

Through discussion of the meaning of division by fractions, the teachers mentioned several concepts that they considered as pieces of the knowledge package related to the topic: the meaning of whole number multiplication, the concept of division as the inverse of multiplication, models of whole number division, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, etc. Figure 3.2 gives an outline of the relationships among these items.

The learning of mathematical concepts is not a unidirectional journey. Even though the concept of division by fractions is logically built on the previous learning of various concepts, it, in turn, plays a role in reinforcing and deepening that previous learning. For example, work on the meaning

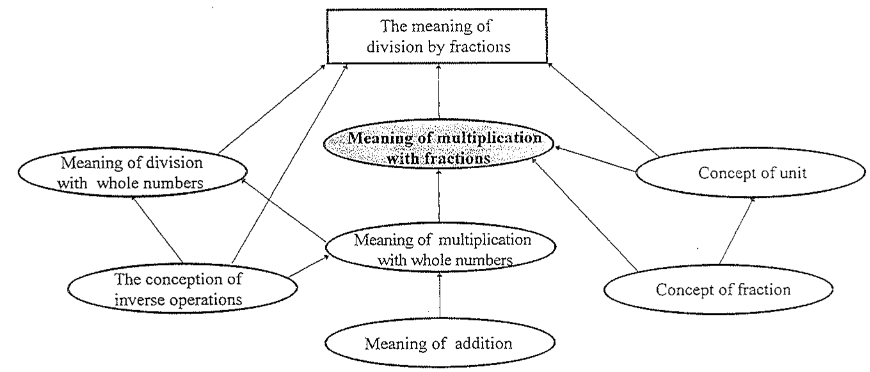

FIG. 3.2. A knowledge package for understanding the meaning of division by fractions.

of division by fractions will intensify previous concepts of rational number multiplication. Similarly, by developing rational number versions of the two division models, one's original understanding of the two whole number models will become more comprehensive:

> This is what is called "gaining new insights through reviewing old ones." The current learning is supported by, but also deepens, the previous learning. The meaning of division by fractions seems complicated because it is built on several concepts. On the other hand, however, it provides a good opportunity for students to deepen their previous learning of these concepts. I am pretty sure that after approaching the meaning and the models of division by fractions, students' previous learning of these supporting concepts will be more comprehensive than before. Learning is a back and forth procedure. (Tr. Sun)

From this perspective, learning is a continual process during which new knowledge is supported by previous knowledge and the previous knowledge is reinforced and deepened by new knowledge.

During the interviews, "the meaning of multiplication with fractions" was considered a key piece of the knowledge package. Most teachers considered multiplication with fractions the "necessary basis" for understanding the meaning of division by fractions:

> The meaning of multiplication with fractions is particularly important because it is where the concepts of division by fractions are derived . . . Given that our students understand very well that multiplying by a fraction means finding a fractional part of a unit, they will follow this logic to understand how the models of its inverse operation work. On the other hand, given that they do not have a clear idea of what multiplication with fractions means, concepts of division by a fraction will be arbitrary for them and very difficult to understand. Therefore, in order to let our students grasp the