Math 143
History of Mathematics

Newton’s Method

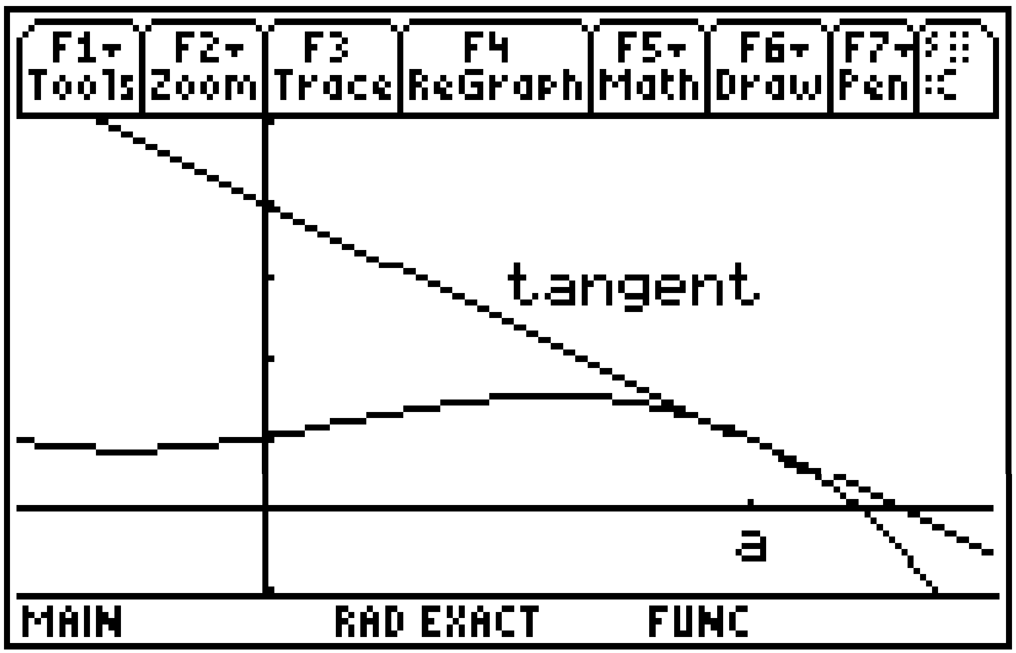
Cal. State, Fresno
Burger

Newton's method is an algorithm which makes use of the derivative of a function to approximate the zeros of the function.

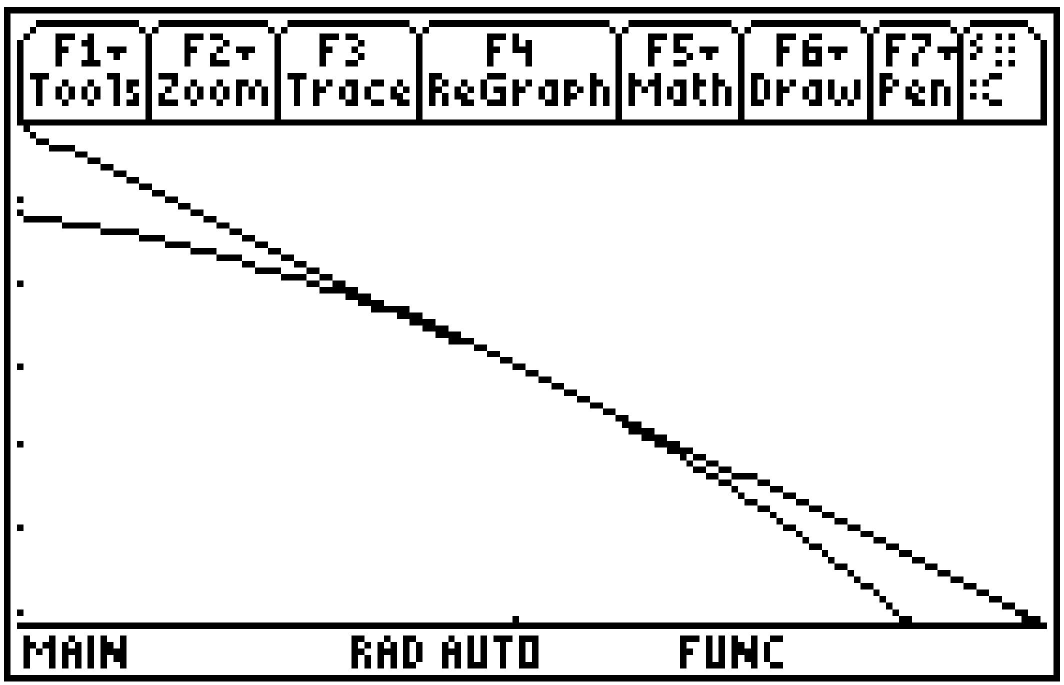
To understand how and why Newton's method works, let's first concentrate on the derivative of a function.

One of the first things you learned about the derivative was that it could be used to find the slope of the tangent to a curve at a specified point. This in turn allowed you to find the equation of that tangent.

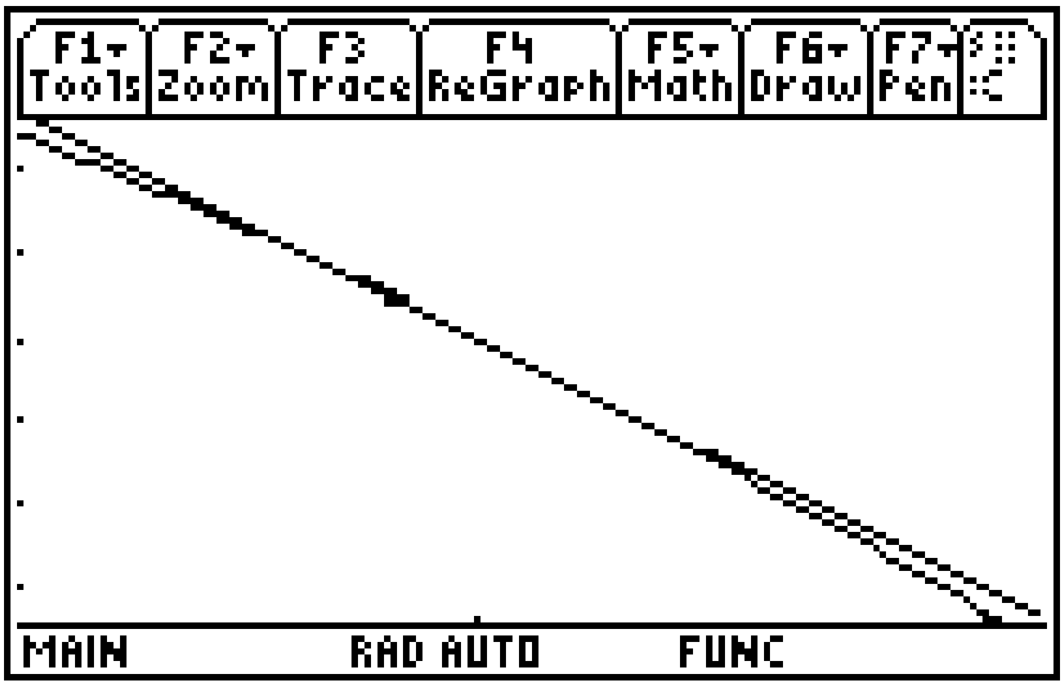
To see this, consider the following figure where we have graphed a curve and its tangent at the point where $x = a$.



When we “zoom in” on the point of tangency we see that around the point of tangency it becomes difficult to distinguish between the graph of the curve and the graph of the tangent.



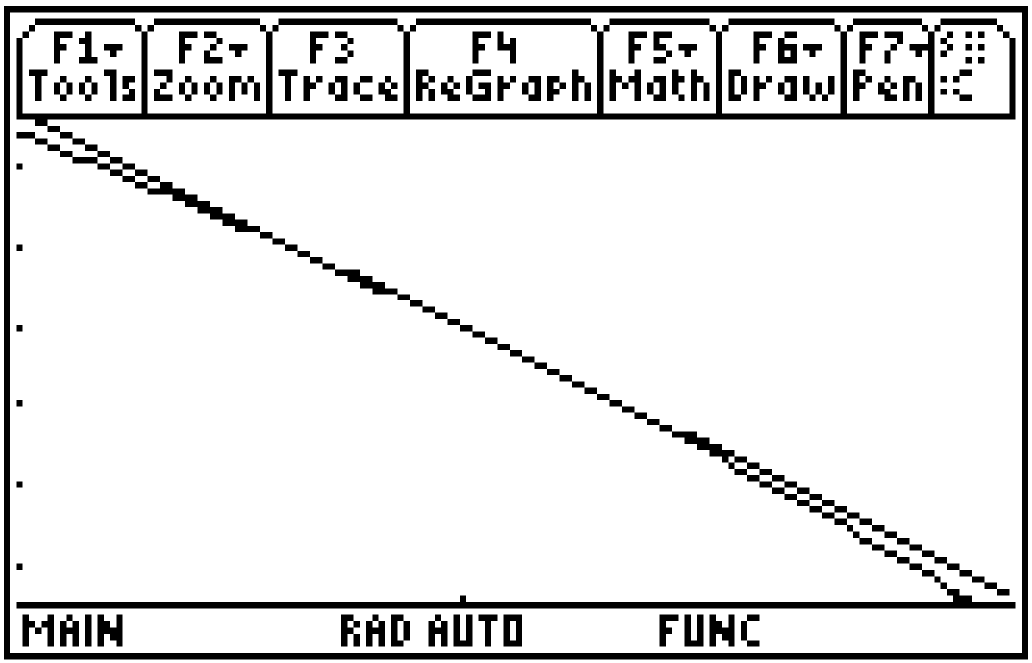
If we “zoom in” again it becomes even more difficult to distinguish between the curve and the tangent.



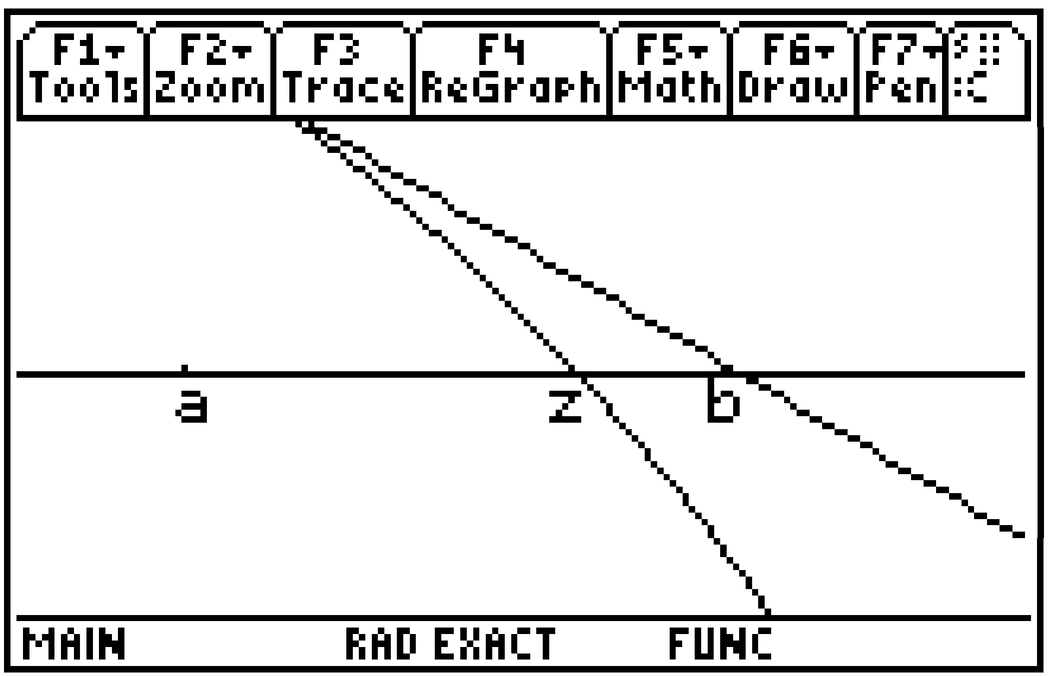
So in the “window” depicted in the previous figure, the points on the tangent can be used to approximate the points on the curve.

But can we use the points on the tangent to approximate the points on the curve which are ***not*** close to the point of tangency? In particular, can we use the zero of the tangent to approximate the zero of a function?

The last figure shows that the zero of the tangent and the zero of the curve are close, but not that close.

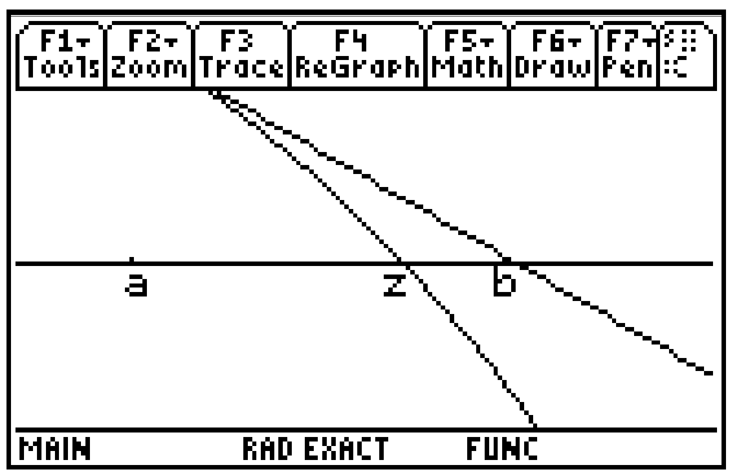


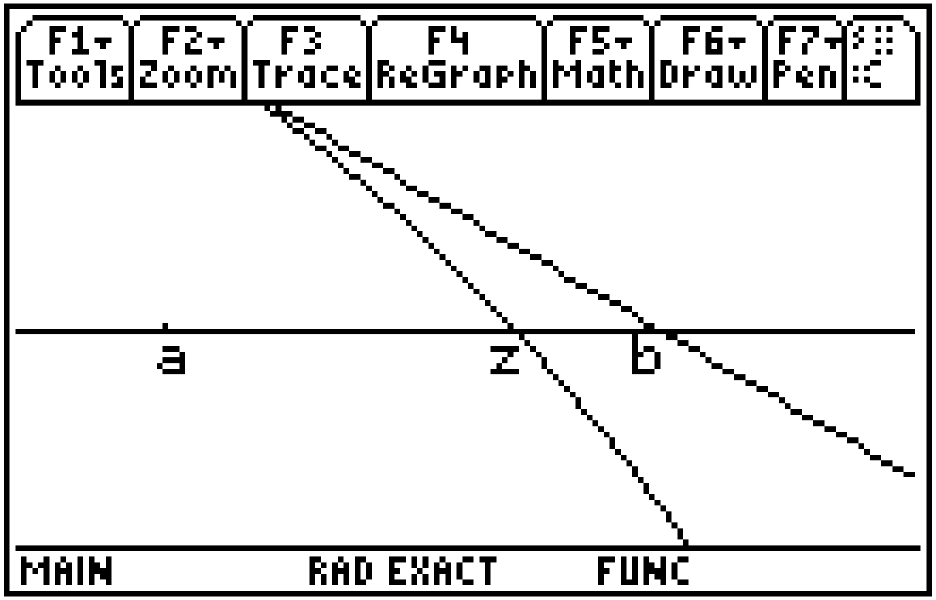
This is pictured below where we have "zoomed in" on these zeroes. In this figure *a* is the *x*-coordinate of the point of tangency, *z* is the zero of the function, and *b* is the zero of the tangent.



In this situation, the zero of the tangent doesn't give a very good approximation of the zero of the function.

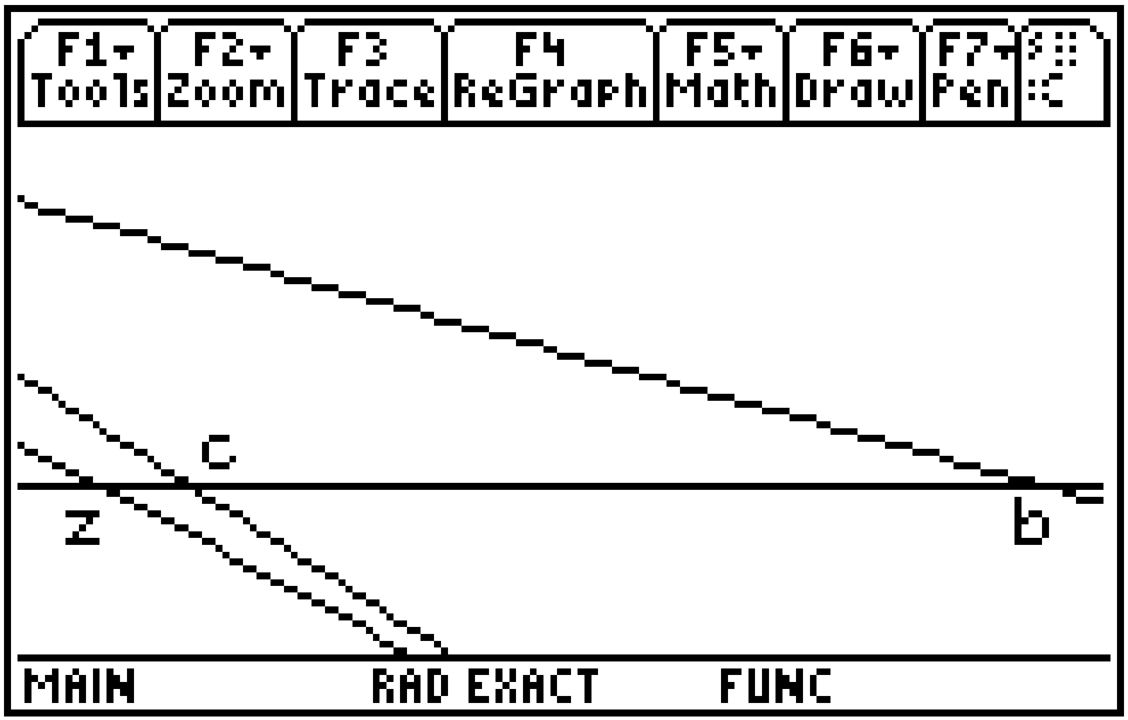
But what if we moved the point of tangency, *a*, closer to the zero of the function?





Since *b* is closer to *z* than *a* is, let's create a tangent to the function at the point where $x = b$.

The result is pictured below where *c* is the zero of the new tangent.



As we can see, c is a much better approximation of z than b was.

And we could probably get an ever better approximation if we continued this process by looking the zero of the tangent at the point where $x = c$.

This is the basis of Newton’s method. To approximate a zero of a function $f(x)$:

1. Pick a value x_1 which is close to the zero of the function.
2. Find the zero x_2 of the tangent to $f(x)$ at the point $(x_1, f(x_1))$.
3. Find the zero x_3 of the tangent to $f(x)$ at the point $(x_2, f(x_2))$.
4. Continue this iterative process.

Each new x_n should be a better approximation of the zero of the function than the previous approximation x_{n-1} .

But what is a formula for x_n?

Since $f'(x_{n-1})$ is the slope of the tangent to $f(x)$ at the point $(x_{n-1},\ f(x_{n-1}))$, the equation of this tangent is

$$y - f(x_{n-1}) = f'(x_{n-1}) \cdot (x - x_{n-1}).$$

And it is easy to show that the zero of this tangent is

$$x = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

But in Newton's method, this zero is denoted by x_n . Hence the iterative formula for Newton's method is:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$