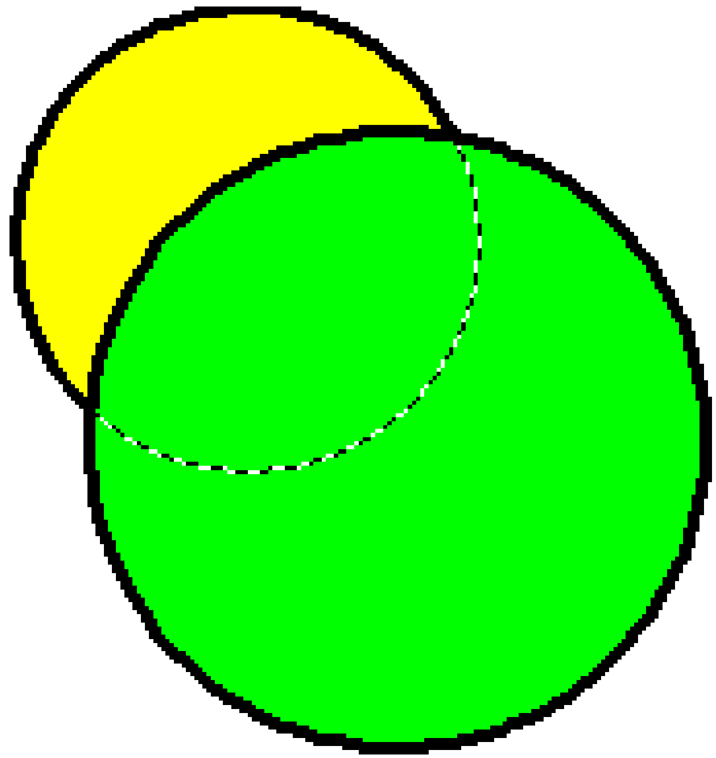
What’s a lune?

A **lune** is a plane figure bounded by two circular arcs.

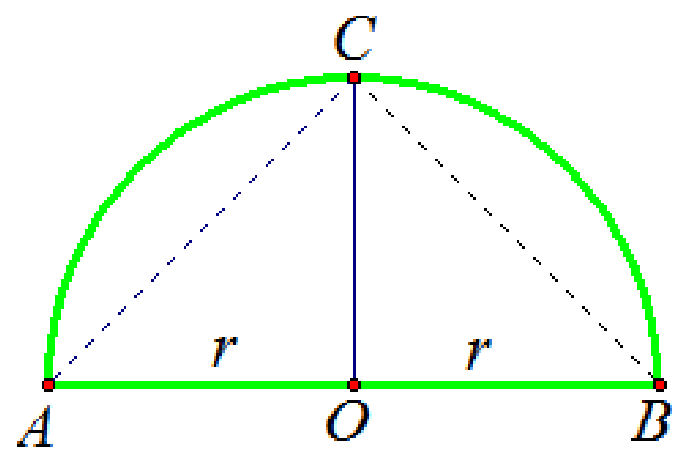
The smaller yellow region in the figure at the right is an example of a lune.



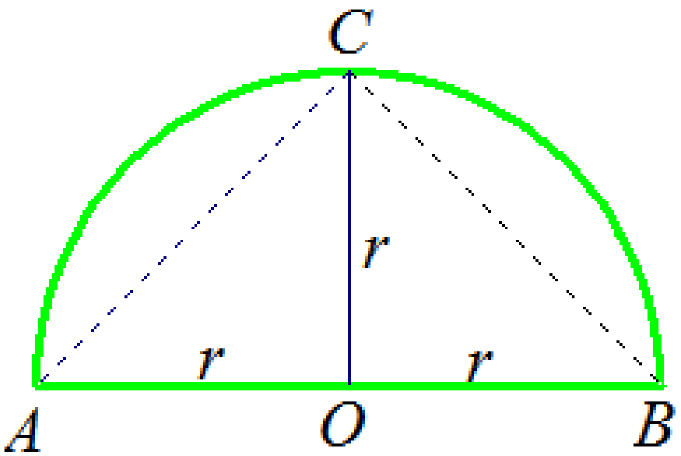
How Hippocrates squared a lune:

Begin with a semicircle having center *O* and radius $\overline{AO} = \overline{OB} = r$.

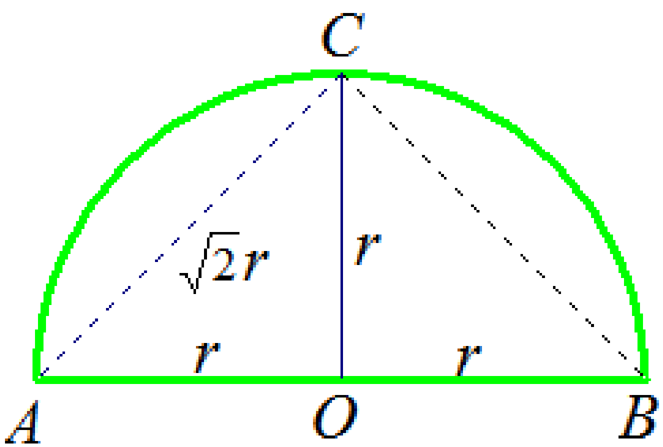
- Construct *OC* perpendicular to *AB* with *C* on the semicircle.
- Construct segments *AC* and *BC*.



Since *OC* is a radius of the semicircle, $\overline{OC} = r.$



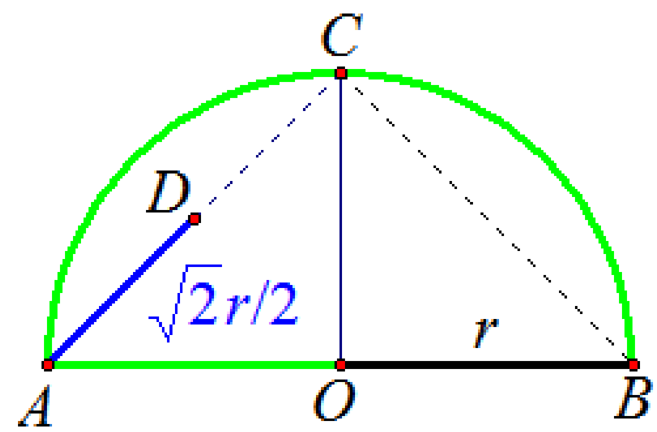
By the Pythagorean theorem, $\overline{AC} = \sqrt{2}\,r.$



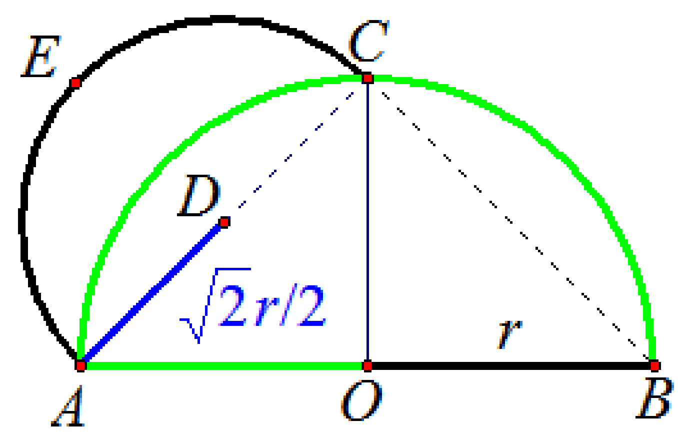
- Construct the midpoint *D* of *AC*.

Since $\overline{AC} = \sqrt{2}\, r,$

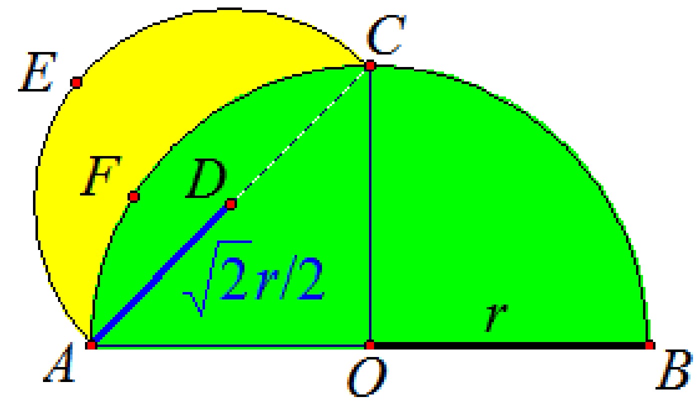
$\overline{AD} = (\sqrt{2}\, r)/2.$



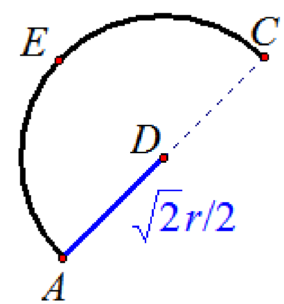
- Construct the semicircle centered at *D* and having radius *AD*, as pictured.



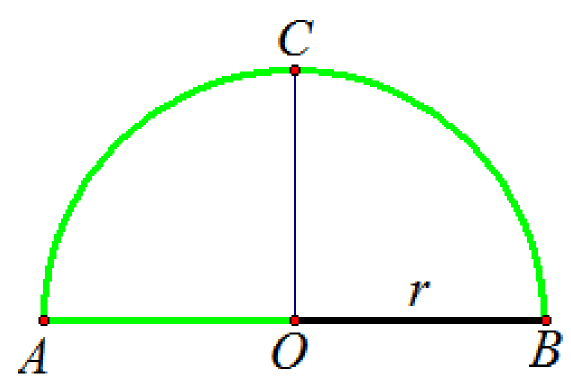
Our goal is to show that the yellow lune *AECF* is squarable.



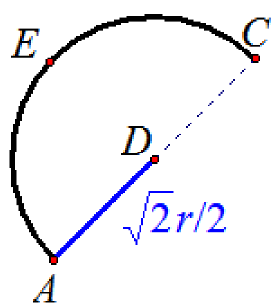
The area of the small semicircle *AEC* is $\dfrac{\pi r^2}{4}$



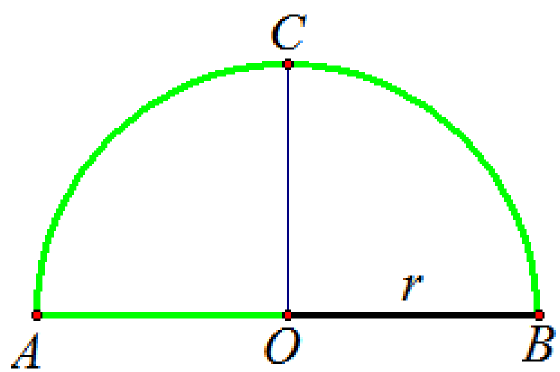
And the area of the large semicircle *ACB* is πr^2.



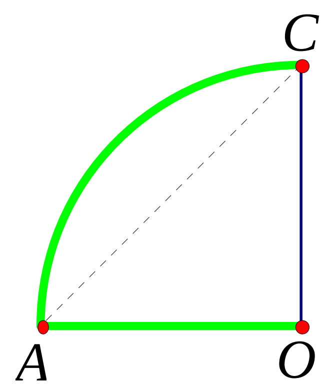
So, it terms of areas,



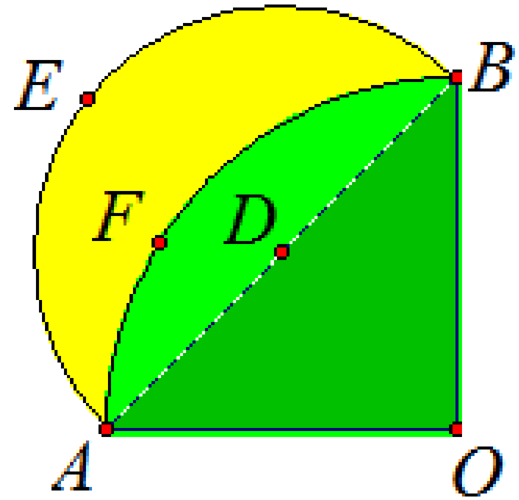
$= \frac{1}{2}$

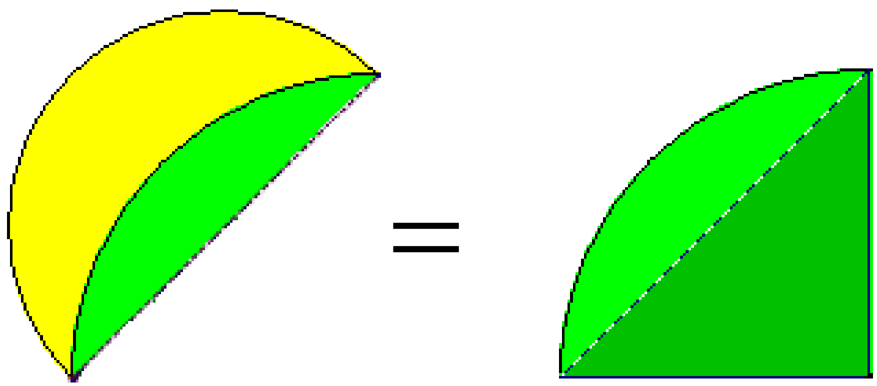


$=$



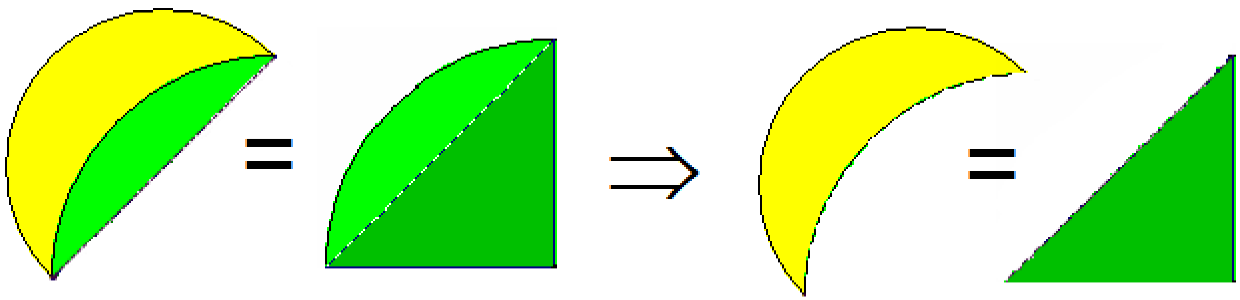
In terms of the first octant of our shaded figure, this says that:





small semicircle = 1/2 large semicircle.

Subtracting the common area shows that the area of the lune is equal to the area of a triangle.



Since a triangle is squarable, as shown in the last lesson, so is Hippocrates' lune.

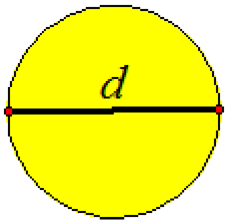
Attempt to square a circle

History reports that Hippocrates also claimed to have squared a circle.

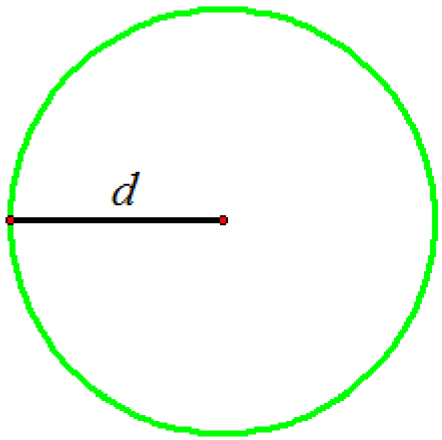
There is no record of his proof.

However, mathematicians speculate that his proof may have been something like what I am about to show you.

Begin with an arbitrary circle having diameter *d*.

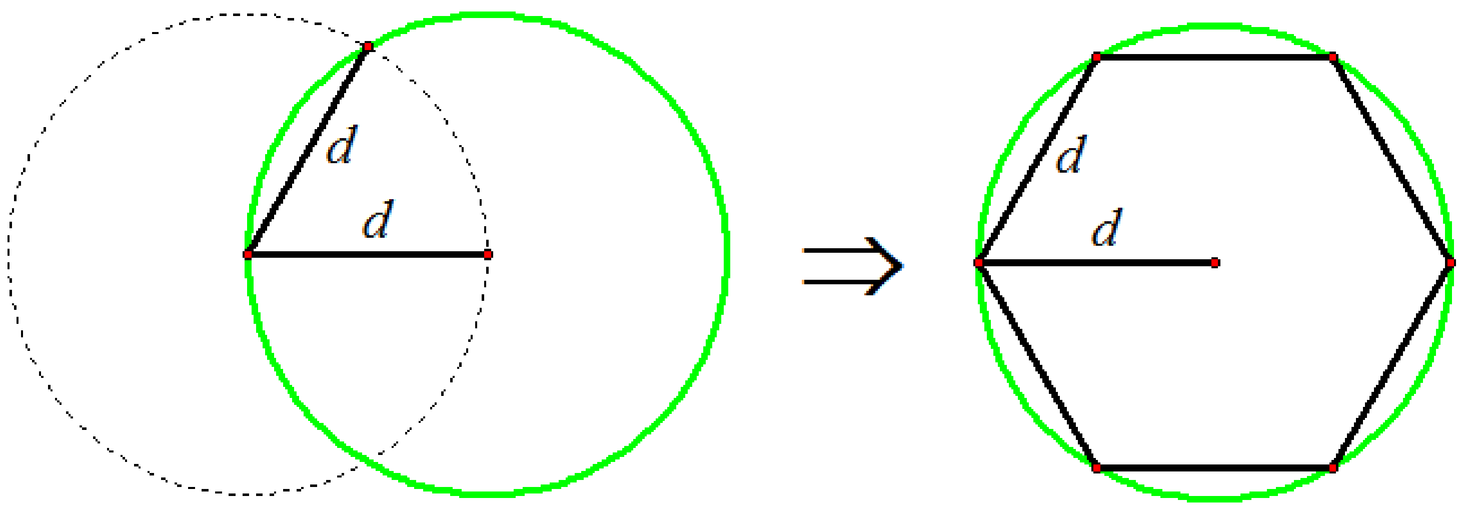


- Construct a circle having radius *d.*



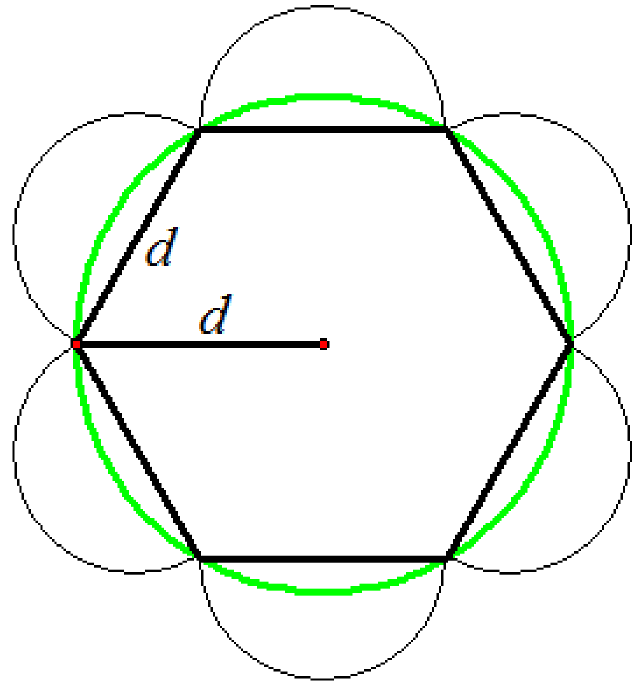
- Inscribe a regular hexagon in that circle.

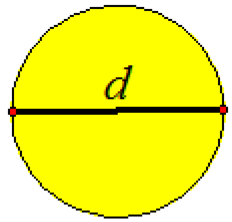
Note: each side of the hexagon will measure *d*.



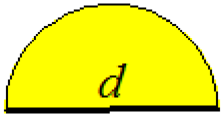
- Construct a semicircle on each side of the inscribed hexagon.

  Note that each semicircle has half the area of the given circle.



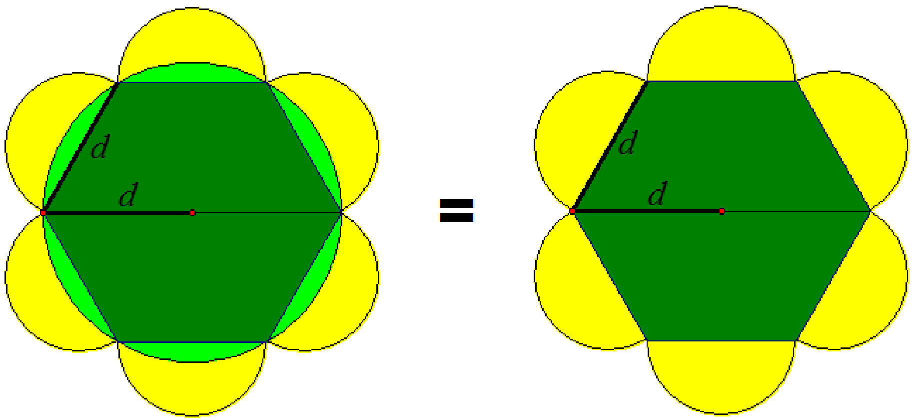


given circle

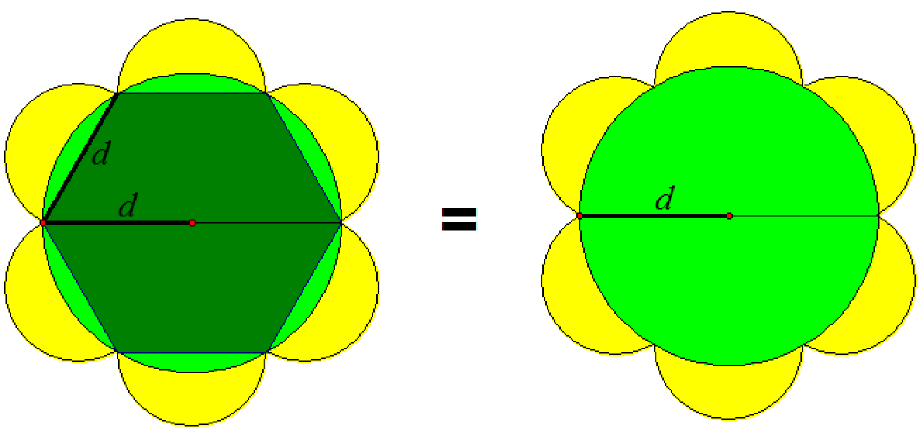


semicircle

The area of our construction can be viewed in two ways:

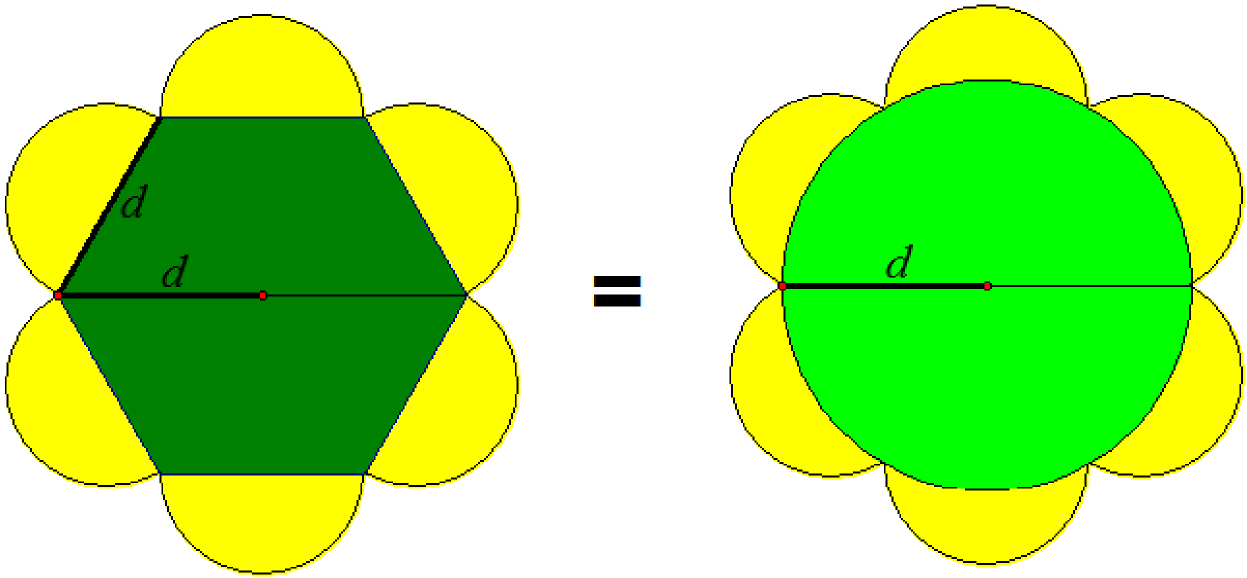


Hexagon plus six semicircles.



Large circle plus six lunes.

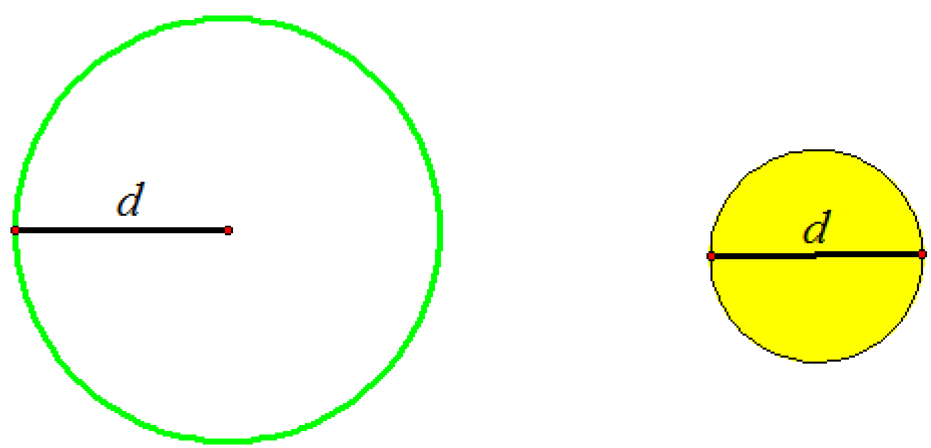
In terms of areas, this gives us:



hexagon + 3 original circles =
large circle + 6 lunes.

But the original circle has diameter d, and thus radius $d/2$, and the large circle has radius d. So

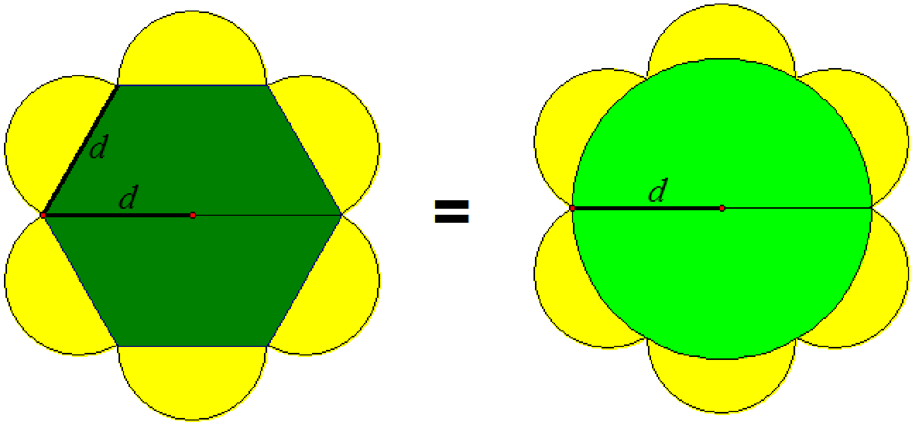
$$\text{Area (large circle)} = \pi d^2 = 4\,(\pi d^2)/4$$
$$= 4\ \text{Area (original circle)}.$$



Since

Area (large circle) = 4 Area (original circle),

in terms of areas,



hexagon + 3 original circles =

4 original circles + 6 lunes.

Simplifying

hexagon + 3 original circles =
4 original circles + 6 lunes

results in:

Area (original circle) =
Area (hexagon) – 6 Area (lune).

Area (original circle) =
Area (hexagon) – 6 Area (lune).

CORRECT?

Area (original circle) =

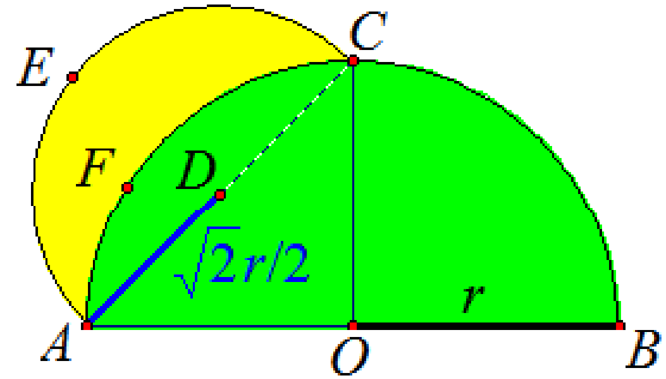
Area (hexagon) – 6 Area (lune)

Hippocrates says you can square a hexagon, you can square a lune, and you can square the difference of two squares. So you can square a circle. Right?

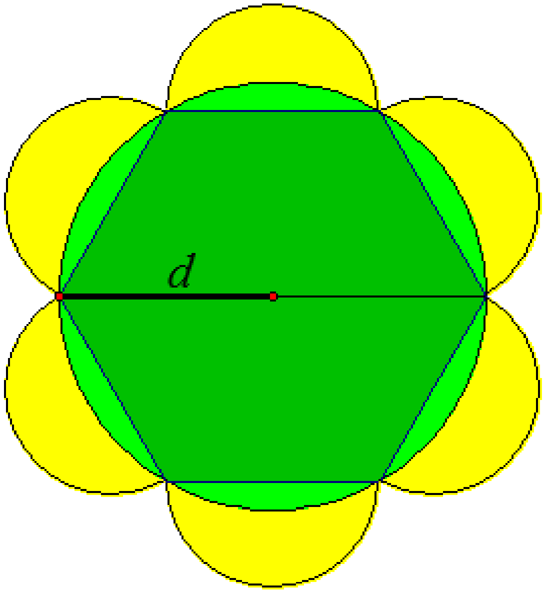
WRONG!

Hippocrates showed that you can square ***a*** lune, but he didn’t show that you could square ***every*** lune.

The lune he squared was a lune constructed on the side of a square inscribed in a circle.



The lune in this bogus squaring of a circle was constructed on the side of a hexagon.



So, can you or can't you square a circle?

The answer is ***NO***, or at least you can't square *every* circle.

This was proved in 1882 by Ferdinand Lindemann. His proof is algebraic, not geometric.