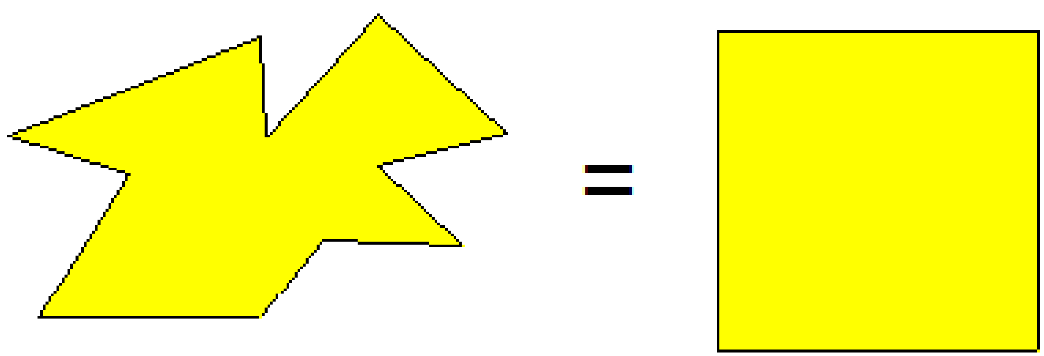
What is quadrature?

The **quadrature** (or squaring) of a plane figure is the construction – *using only straightedge and compass* – of a square having area equal to that of the original plane figure.

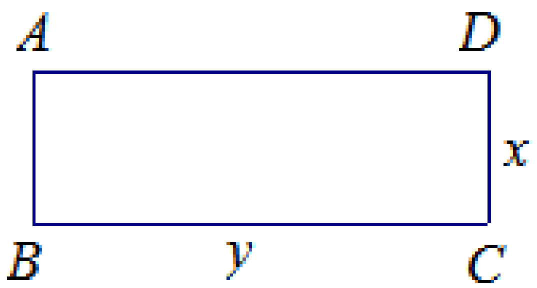
The quadrature problem appealed to the ancient Greeks because finding the area of an irregularly shaped figure is not easy.

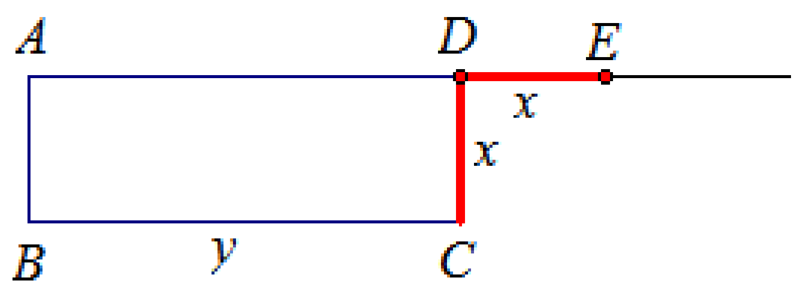
Being able to replace the irregular figure with an equivalent square, would make determining its area a trivial matter.



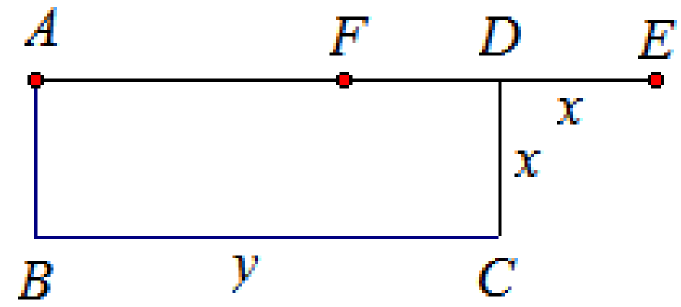
Squaring a rectangle

- Start with a rectangle *ABCD* with sides measuring *x* and *y*, as pictured.
- Construct point *E* on side *AD* extended such that *DE* measures *x.*

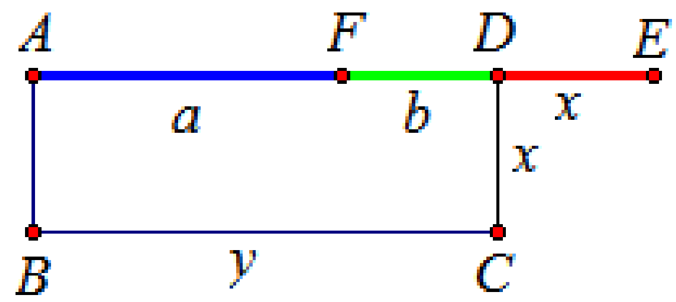




- Construct the midpoint *F* of segment *AE*.



- Denote the measure of *AF* by *a* and the measure of *FD* by *b*.



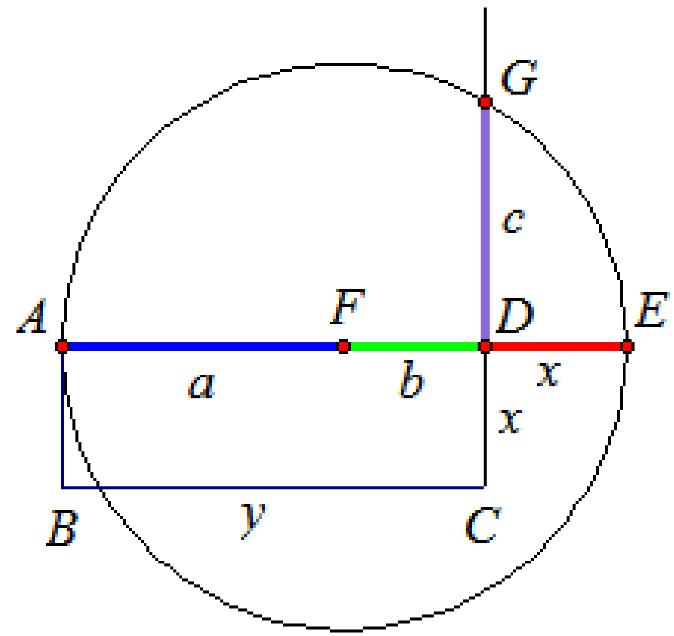
Then $\overline{AF} = \overline{FE} \Rightarrow a = b + x \Rightarrow x = a - b,$

and $\overline{BC} = \overline{AD} \Rightarrow y = a + b.$

- Construct a circle centered at *F* and having radius *a*.
- Let *G* denote the point where this circle intersects the extension of side *DC*, as pictured.
- Denote the measure of *GD* by *c*.

$$x = a - b$$

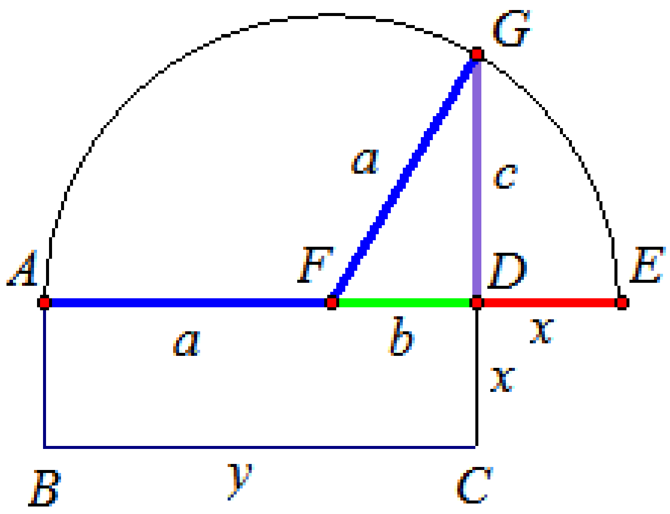
$$y = a + b$$



Since *FG* is a radius of the circle, it has measure *a*.

$$x = a - b$$

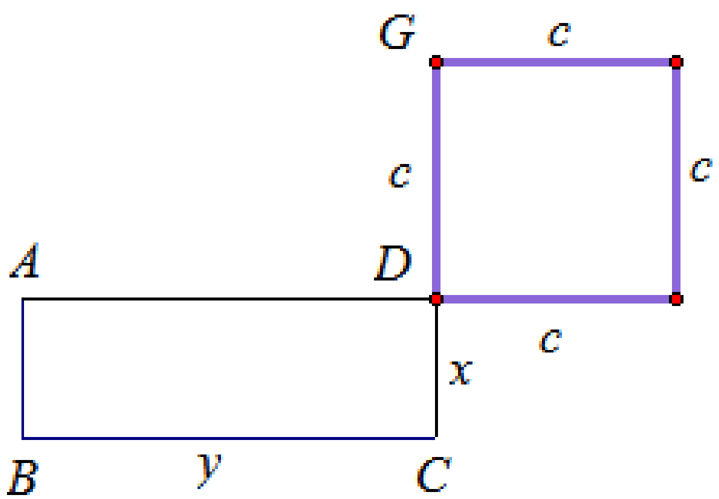
$$y = a + b$$



So by the Pythagorean theorem, $b^2 + c^2 = a^2$.

And thus, $c^2 = a^2 - b^2$.

- Construct a square having side *GD*, as pictured.

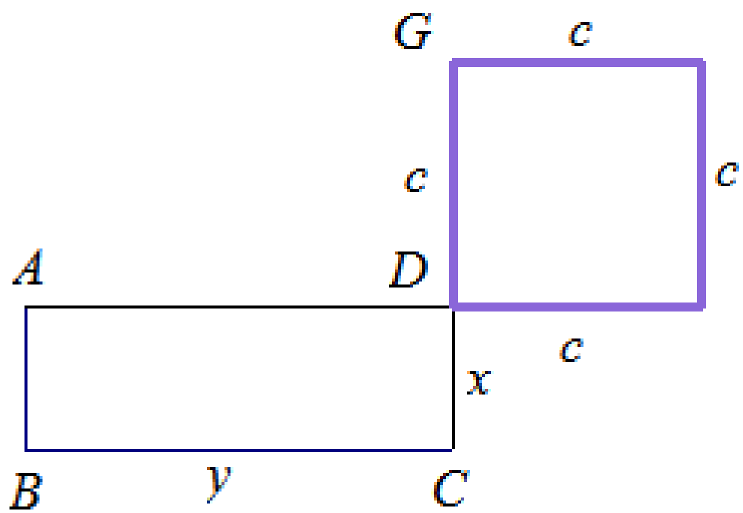


$$x = a - b$$

$$y = a + b$$

$$c^2 = a^2 - b^2$$

Then:



$$x = a - b$$

$$y = a + b$$

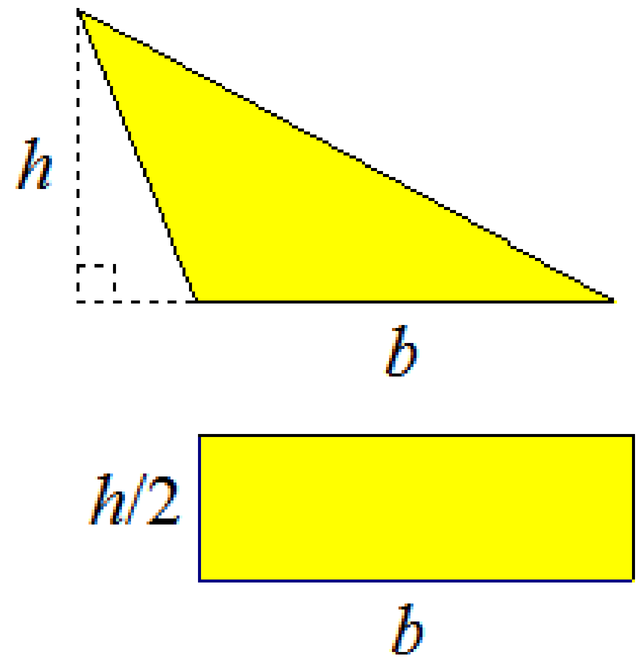
$$c^2 = a^2 - b^2$$

$$\mathrm{A}_{\mathrm{rect}} = xy = (a-b)(a+b) = a^2 - b^2 = c^2 = \mathrm{A}_{\mathrm{sq}}$$

This shows that the ancient Greeks were able to square a rectangle.

Squaring a triangle

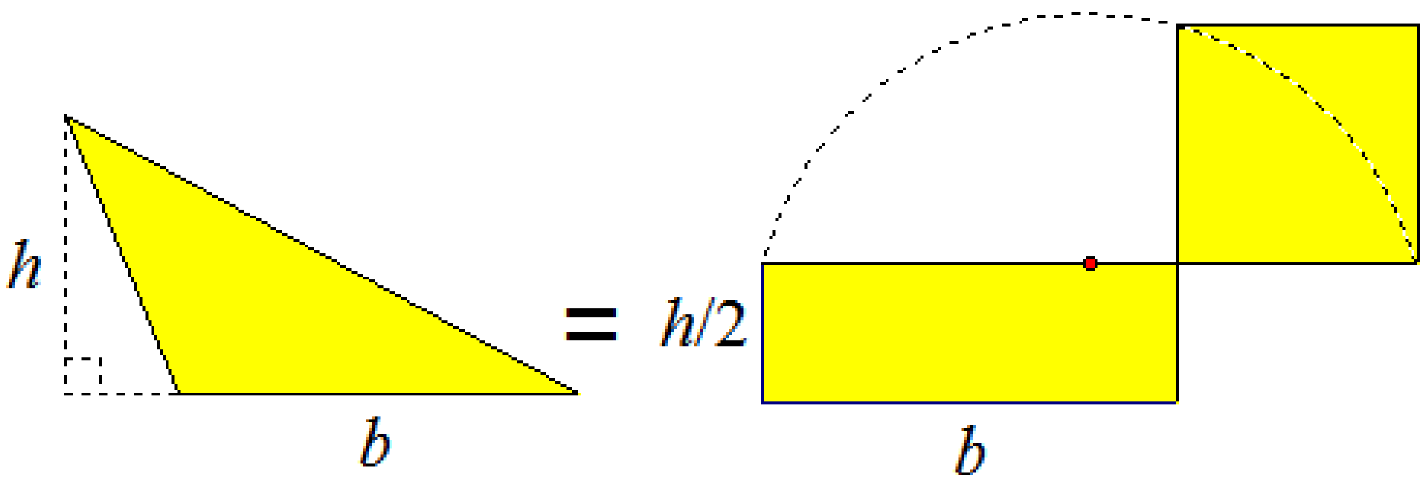
Given a triangle having base measuring *b* and height measuring *h*, construct a rectangle having sides measuring *h*/2 and *b*.



$$A_{\text{triangle}} = \tfrac{1}{2}bh = (\tfrac{1}{2}h)b = A_{\text{rectangle}}$$

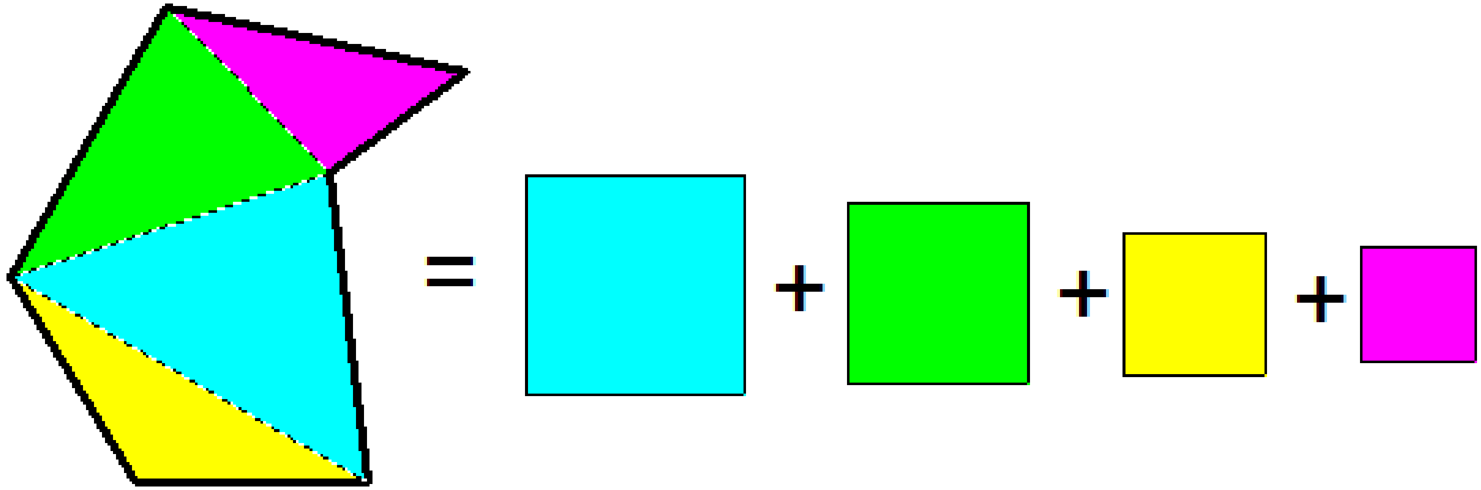
Then square the rectangle.

The area of the triangle is then equal to the area of the square.

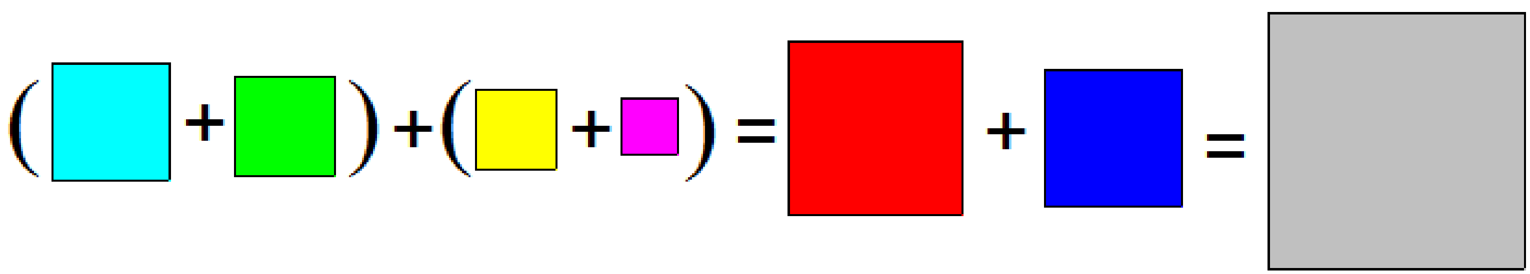


Squaring a polygon

By drawing diagonals, a polygon can be subdivided into triangular regions. And each of these triangles can be squared.

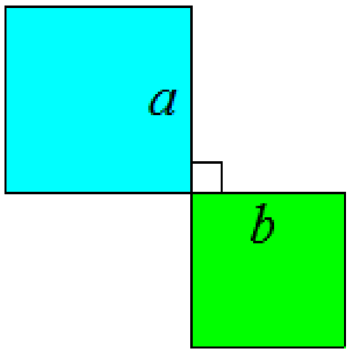


So squaring a polygon boils down to showing that the sum of two squares can be squared.



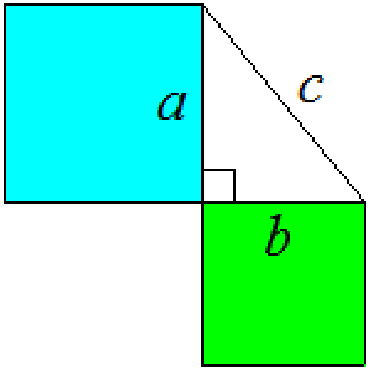
Squaring the sum of two squares

Let the sides of the two squares measure *a* and *b* respectively.

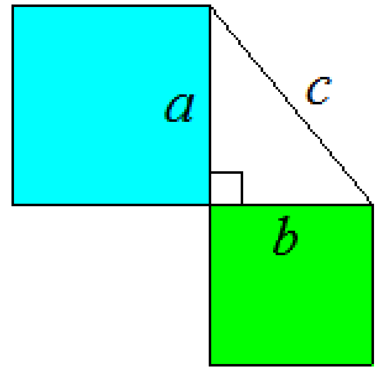


Place these squares so they form a right angle, as pictured.

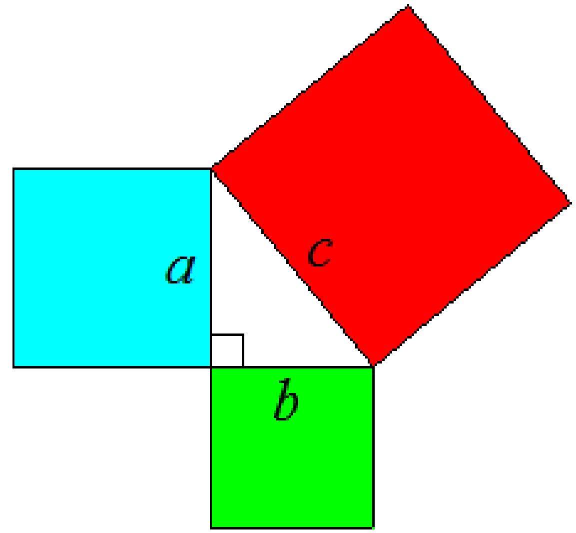
Let *c* denote the measure of the hypotenuse of the right triangle formed by the sides of the two squares.



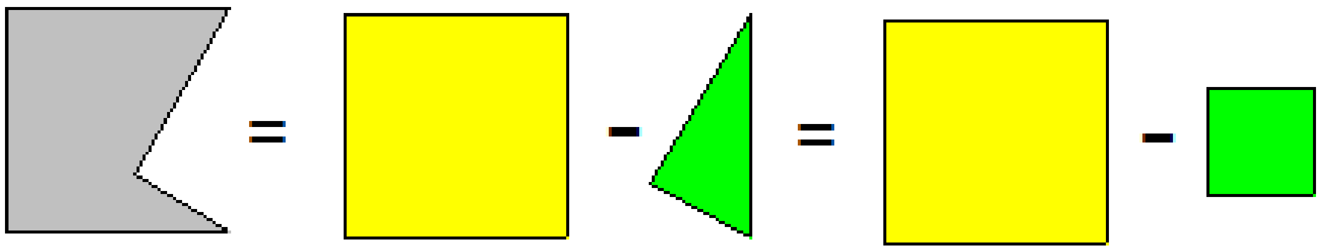
By the Pythagorean theorem, $a^2 + b^2 = c^2$.



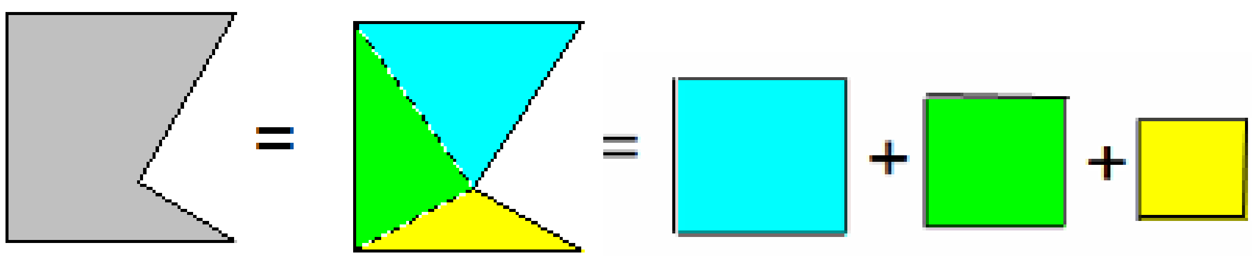
So constructing a square having the hypotenuse of the triangle as a side results in a square whose area is equal to the sum of the two original squares.



Sometimes it’s easier to express the area of a polygon as the difference of squares

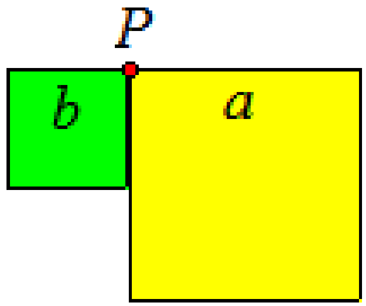


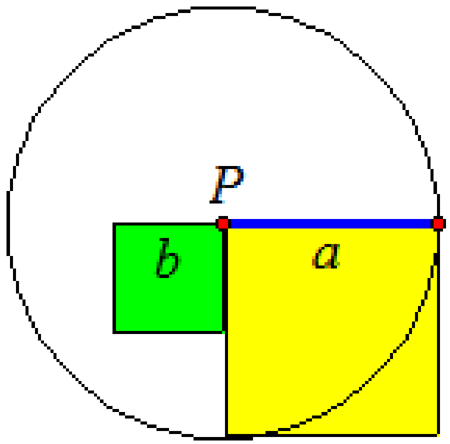
than as the sum of squares.



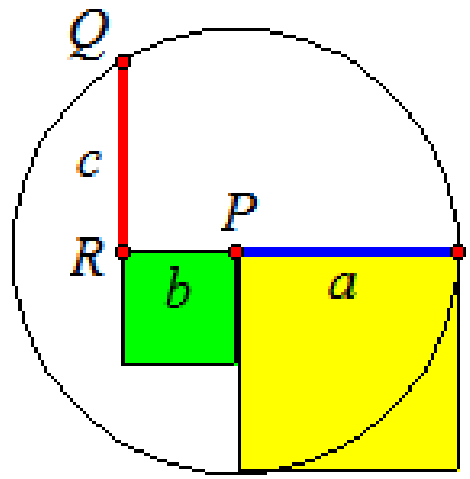
Squaring the difference of two squares

- Let the two squares have sides measuring *a* and *b*, respectively, with *a* > *b*. Place these squares as pictured.
- Construct a circle centered at *P* and having radius *a*.

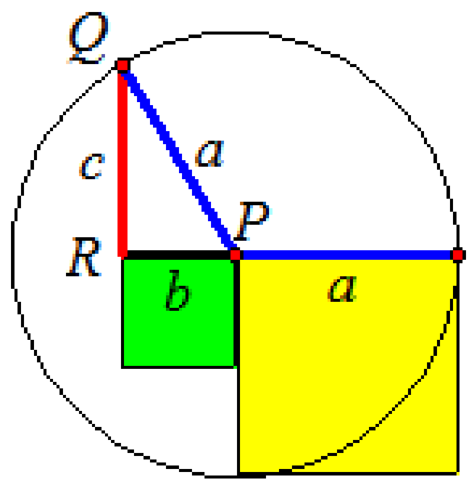




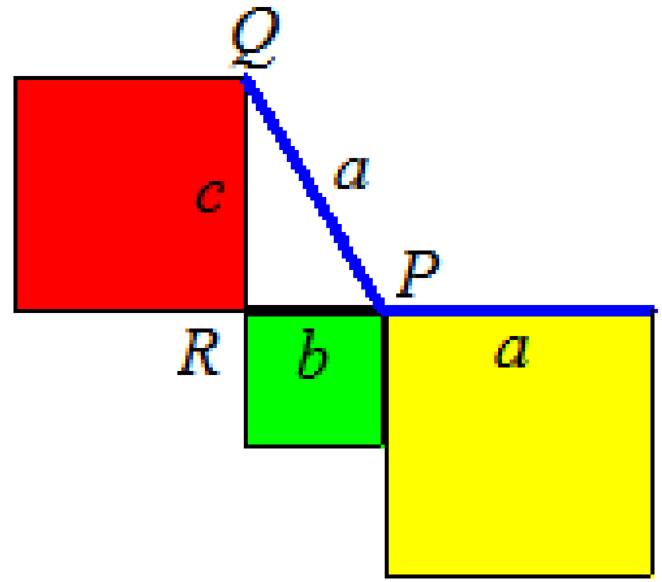
- Construct segment *QR* as pictured and denote it's measure by *c*.



- Construct segment *PQ*. Since *PQ* is a radius of the circle, its measure is *a*.



- Construct a square having side *QR*, as pictured.



Since *PQR* is a right triangle, $a^2 - b^2 = c^2$.

So