1. (10 pts) Find all $x$ such that:

(a) $\left| \frac{x + 5}{-3} \right| \geq 6$

Either $\frac{x + 5}{-3} \geq 6$, or $\frac{x + 5}{-3} \leq -6$. The first inequality simplifies as follows:

$$x + 5 \geq 6 \implies x + 5 \leq -18 \implies x \leq -23.$$

The second inequality simplifies as follows:

$$x + 5 \leq -6 \implies x + 5 \geq 18 \implies x \geq 13.$$

Thus $x \in (-\infty, -23] \cup [13, \infty)$.

(b) $|10x - 4| = 9$

Either $10x - 4 = 9$, or $-(10x - 4) = 9$. The first case gives $x = \frac{13}{10}$, and the second case gives $x = -\frac{1}{2}$.

2. (12 pts) Given $P(-1, 2)$ and $Q(3, 5)$, find the following.

(a) The distance $|PQ|$.

The distance formula says that $D = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. Applying this here, we get

$$D = \sqrt{(5 - 2)^2 + (3 - (-1))^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

(b) The slope of the line through $P$ and $Q$.

The slope is $\frac{y_2 - y_1}{x_2 - x_1}$. Applying this here, we get

$$\frac{5 - 2}{3 - (-1)} = \frac{3}{4}.$$
(c) The line through $P$ parallel to the $x$-axis.

A line parallel to the $x$-axis has slope 0. The $y$-coordinate of $P$ is 2, so the horizontal line through this point is

$$y = 2.$$

(d) The line through $Q$ perpendicular to the line $2x + 6y + 3 = 0$.

Let’s put the line $2x + 6y + 3 = 0$ into slope-intercept form:

$$6y = -2x - 3 \implies y = -\frac{2}{6}x - \frac{3}{6} \implies y = -\frac{1}{3}x - \frac{1}{2}.$$

Thus this line has slope $-\frac{1}{3}$. Any line perpendicular to this line has slope 3 (take the negative reciprocal of $-1/3$).

So we need to find the line through $Q(3, 5)$ which has slope 3:

$$\frac{y - 5}{x - 3} = 3 \implies y - 5 = 3(x - 3) \implies y = 3x - 4.$$

This is the line we want.

3. (4 pts) Do the following.

(a) Convert the angle of $-\frac{8\pi}{5}$ radians to degrees.

$-\frac{8\pi}{5}$ radians is $\frac{-8\times180}{5}$ degrees. This simplifies to $-288$ degrees.

(b) Convert the angle of 150 degrees to radians.

150 degrees $= 150\times\frac{\pi}{180}$ radians. This simplifies to $\frac{5\pi}{6}$ radians.
4. (10 pts) Find the exact values.

(a) \( \sin \left( \frac{4\pi}{8} \right) \)

\[ = \sin \left( \frac{\pi}{2} \right) = 1. \]

(b) \( \tan \left( -\frac{\pi}{3} \right) \)

\[ = \frac{\sin(-\pi/3)}{\cos(-\pi/3)} = \frac{-\sqrt{3}}{2} = -\sqrt{3}. \]

(c) \( \cos^2 \left( \arctan \left( \frac{1}{8} \right) \right) + \sin^2 \left( \arctan \left( \frac{1}{8} \right) \right) \)

\[ \cos^2 \theta + \sin^2 \theta = 1 \text{ for any angle } \theta. \]

(d) \( 2 \sin \left( \frac{\pi}{12} \right) \cos \left( \frac{\pi}{12} \right) \)

For any \( \theta \), \( 2 \sin \theta \cos \theta = \sin(2\theta) \). Thus

\[ 2 \sin(\pi/12) \cos(\pi/12) = \sin(2 \cdot \frac{\pi}{12}) = \sin(\frac{\pi}{6}) = \frac{1}{2}. \]

(e) \( \arccos \left( \cos \left( \frac{\pi}{9} \right) \right) \)

By definition, \( \arccos x \) is the angle in \([0, \pi]\) whose cosine is \( x \). Thus \( \arccos(\cos(\frac{\pi}{9})) \) is the angle in \([0, \pi]\) whose cosine is \( \cos(\frac{\pi}{9}) \). This angle is \( \frac{\pi}{9} \).

Aside: In the above example, the \( \arccos \) and \( \cos \) cancel. However, this does not always happen. Note for example that \( \arccos(\cos(-\frac{\pi}{2})) \) is the angle in \([0, \pi]\) whose cosine is \( \cos(-\frac{\pi}{2}) = 0 \). Because \( -\frac{\pi}{2} \) doesn’t live in \([0, \pi]\), we need to look for another angle whose cosine is 0, namely \( \frac{\pi}{2} \). So \( \arccos(\cos(-\frac{\pi}{2})) = \frac{\pi}{2} \).
5. (9 pts) Find functions whose graphs are obtained from the graph of \( f(x) = x^3 + 1 \) by:

(a) Shifting 6 units to the right.
\[
y = (x - 6)^3 + 1
\]

(b) Vertically stretching by a factor of 10.
\[
y = 10(x^3 + 1)
\]

(c) Reflecting about the \( y \)-axis and then shifting 4 units up.

First reflect about the \( y \)-axis:
\[
y = (-x)^3 + 1 = -x^3 + 1.
\]

Now shift the above result up by 4:
\[
y = -x^3 + 1 + 4 = -x^3 + 5.
\]
6. (10 pts) Do the following.

(a) Let \( f(x) = \ln x \) and \( g(x) = 1 - x^2 \). Find \( f \circ g \) and \( g \circ f \) and their domains.

\[
f \circ g(x) = f(g(x)) = f(1 - x^2) = \ln(1 - x^2).
\]

To find the domain of the above function, note that \( \ln \) must take a positive input. So we need

\[
1 - x^2 > 0,
\]

or

\[
(1 - x)(1 + x) > 0.
\]

The roots are \( x = 1 \), and \( x = -1 \), so there are three intervals to consider: \( (-\infty, -1) \), \( (-1, 1) \), and \( (1, \infty) \). We just have to plug in a test number from each interval to see which intervals to include in the solution. If we plug in \( x = 2 \), we get a negative number. The same happens for \( x = -2 \). Thus we can discard the intervals \( (-\infty, -1) \) and \( (1, \infty) \) from consideration. Plugging in the test number \( x = 0 \), we get 1, which is positive. Hence the solution set of the inequality is the open interval \( x \in (-1, 1) \). This is the domain of \( f \circ g \). (Equivalently, \(-1 < x < 1\).)

Now we solve \( g \circ f \):

\[
g \circ f(x) = g(f(x)) = g(\ln x) = 1 - (\ln x)^2.
\]

In order to plug in a value for \( x \) here, it must be greater than zero, so the domain is \( x \in (0, \infty) \), i.e. all \( x > 0 \).

(b) Find \( f \) and \( g \) such that \( f \circ g = \tan(\sqrt{x}) \).

There are lots of possible solutions here. The most natural one is \( g(x) = \sqrt{x} \) and \( f(x) = \tan x \). Then

\[
f \circ g(x) = f(g(x)) = f(\sqrt{x}) = \tan(\sqrt{x}).
\]

Another possibility is to take \( f(x) = \tan(\sqrt{x}) \) and \( g(x) = x \). Or even \( f(x) = x \) and \( g(x) = \tan(\sqrt{x}) \). (Nobody did this on the test though!)
7. **(10 pts)** Find the inverse function for \( f(x) = \frac{x + 1}{2x + 5} \).

Let \( y = f^{-1}(x) \). Then \( f(y) = x \), i.e.

\[
\frac{y + 1}{2y + 5} = x.
\]

We need to solve for \( y \):

\[
y + 1 = x(2y + 5) = 2xy + 5x.
\]

Putting all the \( y \) terms on the left, we get:

\[
y - 2xy = 5x - 1.
\]

Now factor \( y \) out of the left:

\[
y(1 - 2x) = 5x - 1.
\]

Dividing, we get

\[
f^{-1}(x) = y = \frac{5x - 1}{1 - 2x}.
\]

8. **(10 pts)** Solve the equations given below.

(a) \( 5^x + 5^{x+1} = 10 \)

\[
5^x + 5^{x+1} = 10
\]

\[
5^x(1 + 5) = 10
\]

\[
5^x = \frac{10}{6} = \frac{5}{3}
\]

\[
x = \log_5\left(\frac{5}{3}\right).
\]

(b) \( \ln(x^2) - 9 = \ln x \)

\[
\ln(x^2) - \ln x = 9
\]

\[
\ln \frac{x^2}{x} = 9
\]

\[
\ln x = 9
\]

\[
x = e^9.
\]

9. **(10 pts)** A ladybug is crawling on the \( y \)-axis. If her position at time \( t \) is given by \( y = 2t^2 + 1 \), find her average velocity between \( t = 1 \) and \( t = 3 \) seconds.

Average velocity = Average rate of change of position = \( \frac{p(3) - p(1)}{3 - 1} \)

\[
= \frac{(2(3)^2+1) - (2(1)^2+1)}{2} = \frac{16}{2} = 8.
\]
10. (15 pts) Evaluate each of the following limits. For infinite limits, specify $\infty$ or $-\infty$.

(a) $\lim_{x \to 3} \frac{x^2 + 1}{(x + 2)(x + 6)}$

$$= \frac{10}{(5)(9)} = \frac{2}{9}.$$

(b) $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$

$$= \lim_{x \to 0} \frac{(1 + x) - (1 - x)}{x(\sqrt{1 + x} + \sqrt{1 - x})} = \lim_{x \to 0} \frac{2x}{x(\sqrt{1 + x} + \sqrt{1 - x})}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{1 + x} + \sqrt{1 - x}} = \frac{2}{1 + 1} = 1.$$

(c) $\lim_{x \to 0} |x| \cos \left( \frac{\pi}{x} \right)$

We know that $-1 \leq \cos \left( \frac{\pi}{x} \right) \leq 1$. We can multiply this through by $|x|$ without reversing inequalities since $|x| \geq 0$:

$$-|x| \leq |x| \cos \left( \frac{\pi}{x} \right) \leq |x|.$$

The function on the left and the function on the right both go to 0 as $x \to 0$. Therefore by the squeeze theorem,

$$\lim_{x \to 0} |x| \cos \left( \frac{\pi}{x} \right) = 0.$$

(d) $\lim_{x \to 5^-} \frac{4}{x - 5}$ When $x < 5$ and $x$ is very close to 5, this expression looks like

$$\frac{4}{\text{small negative}},$$

so in the limit, we get $-\infty$.

(e) $\lim_{x \to 7} \frac{\frac{1}{x} - \frac{1}{7}}{x - 7} = \lim_{x \to 7} \frac{\frac{7}{x} - \frac{x}{7x}}{x - 7} = \lim_{x \to 7} \frac{7 - x}{7x} \frac{1}{x - 7} = \lim_{x \to 7} \frac{-1}{7x} = -1/49.$