MATH 141
MIDTERM EXAM II
November 6, 2001

1. **(10 points)** Let $f(x) = x^2 + 2x$.

   (a) Find $f'(x)$ using the definition of the derivative.

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h} =$$
   $$= \lim_{h\to 0} \frac{2xh + h^2 + 2h}{h} = \lim_{h\to 0}(2x + h + 2) = 2x + 2$$

   (b) Find x at which the tangent line to the graph of $f(x)$ is horizontal.

   The tangent line is horizontal when $f'(x) = 0$.
   $2x + 2 = 0$
   $x = -1$

2. **(8 points)** Find an equation of the tangent line to the graph of $f(x) = \dfrac{e^x}{x^2}$ at $(1, e)$.

   The slope of the tangent line is equal to $f'(1)$.
   $$f'(x) = \frac{e^x x^2 - e^x 2x}{(x^2)^2} = \frac{e^x x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}$$
   $f'(1) = -e$
   Therefore the point-slope equation is $y - e = -e(x-1)$, or
   $y = -ex + 2e$

3. **(6 points)** Compute the following limits.

   (a) $\lim_{x\to 0} \sin(x + \sin x) = \sin(0 + \sin 0) = \sin(0) = 0$

   (b) $\lim_{x\to \pi} \sqrt{2 + \cos x} = \sqrt{2 + \cos(\pi)} = \sqrt{2 - 1} = 1$

4. **(12 points)** Compute the following limits.

   (a) $$\lim_{x\to\infty} \frac{x^3 + 2x - 1}{2x^3 - 3x + 4} = \lim_{x\to\infty} \frac{1 + \frac{2}{x^2} - \frac{1}{x^3}}{2 - \frac{3}{x^2} + \frac{4}{x^3}} = \frac{1}{2}$$

   (b) $$\lim_{x\to\infty} \frac{2x}{\sqrt[4]{x^4 + 3}} = \lim_{x\to\infty} \frac{2x}{\left(\frac{\sqrt[4]{x^4+3}}{x}\right)} = \lim_{x\to\infty} \frac{2x}{\left(\frac{\sqrt[4]{x^4+3}}{\sqrt[4]{x^4}}\right)} = \lim_{x\to\infty} \frac{2x}{\sqrt[4]{1 + \frac{3}{x^4}}} =$$
   $$= \frac{2}{\sqrt[4]{1}} = 2$$

   (c) $$\lim_{x\to -\infty} \frac{3x^3 + 2x}{x^2 + 1} = \lim_{x\to -\infty} \frac{3x + \frac{2}{x}}{1 + \frac{1}{x^2}} = \frac{-\infty}{1} = -\infty$$

(d) $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right) = \lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2+2x}\right)\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}} =$

$= \lim_{x\to-\infty}\frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty}\frac{-2}{1-\frac{\sqrt{x^2+2x}}{x}} =$

$= \lim_{x\to-\infty}\frac{-2}{1+\frac{\sqrt{x^2+2x}}{\sqrt{x^2}}} = \lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+\frac{2}{x}}} = \frac{-2}{1+\sqrt{1}} = -1$

5. **(16 points)** Differentiate the following functions.

(a) $\left[\left(\sin\frac{\pi}{6}\right)^3\right]' = 0$ because $\left(\sin\frac{\pi}{6}\right)^3 = \text{constant}$

(b) $\left[e^x\left(x^2+\frac{1}{\sqrt[7]{x}}\right)\right]' = e^x\left(x^2+x^{-\frac{1}{7}}\right) + e^x\left(2x-\frac{1}{7}x^{-\frac{8}{7}}\right) =$

$= e^x\left(x^2+x^{-\frac{1}{7}}+2x-\frac{1}{7}x^{-\frac{8}{7}}\right)$

(c) $\left[\frac{x^3+17}{1+\frac{1}{x}}\right]' = \frac{3x^2(1+\frac{1}{x})-(x^3+17)(-\frac{1}{x^2})}{(1+\frac{1}{x})^2} =$

$= \frac{3x^2+3x+x+\frac{17}{x^2}}{(1+\frac{1}{x})^2} = \frac{3x^2+4x+\frac{17}{x^2}}{(1+\frac{1}{x})^2}$

(d) $\left[\frac{x^2-2}{\sqrt{x}}\right]' = \left[x^{\frac{3}{2}}-2x^{-\frac{1}{2}}\right]' = \frac{3}{2}x^{\frac{1}{2}}-2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}}+x^{-\frac{3}{2}}$

6. **(12 points)** After kicking a ball up a steep hill John waits for it to roll back down. Assuming that its distance from John is given by $f(t) = -2t^2+8t$, answer the following questions.

(a) What is the velocity of the ball at time t?

$v(t) = f'(t) = -4t+8$

(b) At what time is the velocity of the ball zero?

$v(t) = 0 \;\Rightarrow\; -4t+8=0 \;\Rightarrow\; t=2$

(c) How far from John is the ball at a moment when it turns around?

The ball turns around at the moment when its velocity is equal to 0 $\Rightarrow$ $f(2) = -2\cdot 2^2+8\cdot 2 = 8$

(d) At what time does the ball come back to John?

The ball comes back to John when $f(t)=0$.
$-2t^2+8t=0 \;\Rightarrow\; 2t(-t+4)=0$
There are two solutions: $t=0$ is the initial moment, and
$t=4$ is when the ball comes back to John.

7. **(8 points)** Assuming that the amount of bacteria at time t is given by $f(t) = 2e^t + 3t^2 + 10$, find the rate of growth of the bacterial colony when $t = 10$.

The rate of growth is $f'(t) = 2e^t + 6t$.
$f'(10) = 2e^{10} + 60$

8. **(10 points)** Let f be a function defined as follows:

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases}$$

(a) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2$
$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + x) = 2$

(b) Is f continuous on $\mathbb{R}$? Why?

$f(x)$ is continuous at 1 because $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$, and it is continuous at all other points because $x^2 + x$ and $x + 1$ are continuous on $\mathbb{R}$.

(c) Is f differentiable on $\mathbb{R}$? Why?

Clearly, $f(x)$ is differentiable at all points except possibly at 1. It is differentiable at 1 if and only if the slopes of $x^2 + x$ and $x + 1$ agree at 1 (so that the graph is smooth at 1). The derivatives of these functions are $2x + 1$ and 1, so the slopes at 1 are 3 and 1. They are different, therefore $f(x)$ is not differentiable at 1.

9. **(6 points)** Use the Intermediate Value Theorem to show that the equation $x^4 - x - 1 = 0$ has a root in the closed interval $[1, 2]$.

Let $f(x) = x^4 - x - 1$.
$f(1) = -1 < 0$ and $f(2) = 13 > 0$, therefore by the intermediate value theorem there is a point c in $[1, 2]$ s.t. $f(c) = 0$, i.e. $x^4 - x - 1 = 0$ has a root in $[1, 2]$.

10. **(12 points)** Find the horizontal and the vertical asymptotes of $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$.

$$\lim_{x \to +\infty} \frac{x^2 + 1}{x^2 - 1} = \lim_{x \to +\infty} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 1, \qquad \lim_{x \to -\infty} \frac{x^2 + 1}{x^2 - 1} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 1$$

Therefore, $y = 1$ is the only horizontal asymptote.

$f(x)$ is not defined at 1 and -1 (because the denominator is 0 at these points), so it may have vertical asymptotes at 1 and -1. Let's check:

$$\lim_{x \to 1^+} \frac{x^2 + 1}{x^2 - 1} = \lim_{x \to 1^+} \frac{x^2 + 1}{(x - 1)(x + 1)} = +\infty$$

$$\lim_{x \to -1^+} \frac{x^2 + 1}{x^2 - 1} = \lim_{x \to 1^+} \frac{x^2 + 1}{(x - 1)(x + 1)} = -\infty$$

Therefore, $x = 1$ and $x = -1$ are vertical asymptotes.