MATH 141

FINAL EXAM

December 16, 2000

- **No calculators are allowed on this exam.**
- **Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Please put your simplified final answers in the spaces provided.**

Part A

1. **(20pts)** Let $f(x) = x^3 - 3x$.
   (a) Find the tangent line to the graph of $f(x)$ where $x = 2$.
   (b) Find the secant line to the graph of $f(x)$ over the interval $[-2, 4]$.
   (c) The Mean Value Theorem applied to f over the interval $[-2, 4]$ implies that there is a number $c \in (-2, 4)$ such that $f'(c)$ equals to the slope of the above secant. What is c?
   (d) Find the line passing through the point $(2, 2)$ perpendicular to the above secant.
2. **(20pts)** Answer each of the following questions:
   (a) Let $f(t) = \dfrac{\sqrt{t}}{1+t}$. What is $f'(1)$?
   (b) If $y = e^{x\sqrt{2}}$, what is $\dfrac{d^2y}{dx^2}$?
   (c) If $f(\theta) = \sin(\theta)$, what is $f^{(65)}(\theta)$?
   (d) What is $\dfrac{d}{dx}(e^{\sin\sqrt{\pi}} + \ln(2))$?

3. **(10pts)** Let $\theta \in (0, \frac{\pi}{2})$ be an angle such that $\cot(\theta) = \frac{1}{2}$.

   (a) What is $\tan(\theta)$?

   (b) What is $\sec(\theta)$?

4. **(20pts)** Differentiate each of the following functions:

   (a) $(x^2 + x)^{11}$

   (b) $e^x \tan(x)$

   (c) $\dfrac{\sin x}{(x+2)^2}$

   (d) $\sin(e^{x^2})$

5. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

   (a) $\lim\limits_{x \to 3^+} \dfrac{\sqrt{x} - \sqrt{3}}{x - 3}$

   (b) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^2 + x - 2}$

6. **(20pts)** Let f be a function defined as follows:

$$f(x) = \begin{cases} x^2 - 2x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^3 - 2x & \text{if } x > 0 \end{cases}$$

   Because $f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$, function f is continuous at 0 as well as at all other numbers. Recall that:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0},$$

   provided that this limit exists.

   (a) Evaluate the above limit as $x \to 0^+$.

   (b) Evaluate the above limit as $x \to 0^-$.

   (c) Is f differentiable at 0? If it is, what is $f'(0)$?

   (d) What is $f'(x)$ for $x \in (-\infty, 0)$?

End of Part A

Part B

7. **(20pts)** Differentiate each of the following functions:

   (a) $\arctan(3x)$

   (b) $\ln(1+\frac{1}{x})$

   (c) $\ln(2^x x^2)$

   (d) x^{x^2+x}

8. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

   (a) $\lim_{x\to\infty} \frac{(2x+2)^2}{(x+1)^2}$

   (b) $\lim_{x\to-\infty} \frac{x^2+1}{x+1}$

9. **(10pts)** If $y^3 + y^2 x = 3$, find the value of $\frac{dy}{dx}$ at the point $(2,1)$.

10. **(10pts)** Let $f(x) = x^{\frac{2}{3}}$.

    (a) Find the linear approximation for $f(x)$ at 27 (i.e.: an approximation valid for x near 27).

    (b) Use the above to calculate $(27.003)^{\frac{2}{3}}$. Calculate your answer to 5 decimal places.

11. **(10pts)** Air is pumped into a spherical balloon at a rate of $10\,\text{cm}^3/\text{min}$ (recall that the volume and the surface area of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$, respectively).

    (a) What is the rate of change of the radius (in cm/min) at a moment when $r = 9\,\text{cm}$?

    (b) What is the rate of change of the area (in cm^2/min) at the same time?

12. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

    (a) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

    (b) $\lim_{x\to\infty} \sqrt{1+\frac{1}{e^x}}$

13. **(30pts)** Let $f(x) = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x$.

(a) Find all critical numbers of f.

(b) Find all intervals on which f is increasing.

(c) Find all intervals on which f is decreasing.

(d) Find all intervals on which f is concave up.

(e) Find all intervals on which f is concave down.

(f) Find all inflection points of f.