MATH 141

FINAL EXAM ANSWER KEY

December 16, 2000

Part A

1. **(20pts)** Let $f(x) = x^3 - 3x$.

(a) Find the tangent line to the graph of $f(x)$ where $x = 2$.

$$f'(x) = 3x^2 - 3 \implies f'(2) = 9$$

Thus,

$$y - f(2) = 9(x - 2)$$

$$\text{ANSWER:} \quad y - 2 = 9(x - 2)$$

(b) Find the secant line to the graph of $f(x)$ over the interval $[-2, 4]$.

$$m = \frac{f(4) - f(-2)}{4 - (-2)} = 9$$

Thus,

$$y - f(-2) = 9(x - (-2))$$

$$\text{ANSWER:} \quad y + 2 = 9(x + 2)$$

(c) The Mean Value Theorem applied to f over the interval $[-2, 4]$ implies that there is a number $c \in (-2, 4)$ such that $f'(c)$ equals to the slope of the above secant. What is c?

ANSWER: 2

(d) Find the line passing through the point $(2, 2)$ perpendicular to the above secant.

The slope should be $m' = -\frac{1}{m} = -\frac{1}{9}$, so that the line is given by:

$$y - f(2) = -\frac{1}{9}(x - 2).$$

$$\text{ANSWER:} \quad y - 2 = -\frac{1}{9}(x - 2)$$

2. **(20pts)** Answer each of the following questions:

(a) Let $f(t) = \dfrac{\sqrt{t}}{1+t}$. What is $f'(1)$?

$$f'(t) = \frac{\frac{1}{2\sqrt{t}}(1+t) - \sqrt{t}}{(1+t)^2}$$

ANSWER: 0

(b) If $y = e^{x\sqrt{2}}$, what is $\dfrac{d^2y}{dx^2}$?

ANSWER: $2e^{x\sqrt{2}}$

(c) If $f(\theta) = \sin(\theta)$, what is $f^{(65)}(\theta)$?

ANSWER: $\cos(\theta)$

(d) What is $\dfrac{d}{dx}(e^{\sin\sqrt{\pi}} + \ln(2))$?

ANSWER: 0

3. **(10pts)** Let $\theta \in (0, \frac{\pi}{2})$ be an angle such that $\cot(\theta) = \frac{1}{2}$.

(a) What is $\tan(\theta)$?

Recall that $\tan(\theta) = \dfrac{1}{\cot(\theta)}$.

ANSWER: 2

(b) What is $\sec(\theta)$?

$$1 + \tan^2(\theta) = \sec^2(\theta) \implies \sec^2(\theta) = 5$$

Take the positive solution since $\theta \in (0, \frac{\pi}{2})$.

ANSWER: $\sqrt{5}$

4. **(20pts)** Differentiate each of the following functions:

(a) $(x^2 + x)^{11}$

ANSWER: $11(x^2 + x)^{10}(2x + 1)$

(b) $e^x \tan(x)$

ANSWER: $e^x \tan(x) + e^x \sec^2(x)$

(c) $\dfrac{\sin x}{(x+2)^2}$

ANSWER: $\dfrac{(x+2)\cos x - 2\sin x}{(x+2)^3}$

(d) $\sin(e^{x^2})$

ANSWER: $2xe^{x^2}\cos(e^{x^2})$

5. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

(a) $\displaystyle\lim_{x \to 3^+} \frac{\sqrt{x} - \sqrt{3}}{x - 3} = \lim_{x \to 3^+} \frac{\sqrt{x} - \sqrt{3}}{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})} = \lim_{x \to 3^+} \frac{1}{\sqrt{x} + \sqrt{3}}$

ANSWER: $= \dfrac{1}{2\sqrt{3}}$

(b) $$\lim_{x\to1}\frac{x^2-1}{x^2+x-2}=\lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)(x+2)}=\lim_{x\to1}\frac{(x+1)}{(x+2)}$$

ANSWER: $\frac{2}{3}$

6. **(20pts)** Let f be a function defined as follows:

$$f(x)=\begin{cases} x^2-2x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ x^3-2x & \text{if } x>0 \end{cases}$$

Because $f(0)=\lim_{x\to0^+}f(x)=\lim_{x\to0^-}f(x)=0$, function f is continuous at 0 as well as at all other numbers. Recall that:

$$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0},$$

provided that this limit exists.

(a) Evaluate the above limit as $x\to0^+$.

$$\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{x^3-2x}{x}=\lim_{x\to0^+}x^2-2=-2$$

ANSWER: -2

(b) Evaluate the above limit as $x\to0^-$.

$$\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{x^2-2x}{x}=\lim_{x\to0^-}x-2=-2$$

ANSWER: -2

(c) Is f differentiable at 0? If it is, what is $f'(0)$?

ANSWER: YES and $f'(0)=-2$

(d) What is $f'(x)$ for $x\in(-\infty,0)$?

ANSWER: $2x-2$

End of Part A

Part B

7. **(20pts)** Differentiate each of the following functions:

(a) $\arctan(3x)$

ANSWER: $\dfrac{3}{1+9x^2}$

(b) $\ln(1+\dfrac{1}{x})$

ANSWER: $\dfrac{1}{1+\frac{1}{x}}(-\dfrac{1}{x^2}) = -\dfrac{1}{x^2+x}$

(c) $\ln(2^x x^2) = \ln(2^x) + \ln(x^2) = x\ln(2) + 2\ln(x)$

ANSWER: $\ln(2) + \dfrac{2}{x}$

(d) $x^{x^2+x} = e^{(\ln x)(x^2+x)}$

ANSWER: $x^{x^2+x}((x+1) + (\ln x)(2x+1))$

8. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

(a) $\lim\limits_{x\to\infty} \dfrac{(2x+2)^2}{(x+1)^2} = \lim\limits_{x\to\infty} \left(\dfrac{2x+2}{x+1}\right)^2 = \lim\limits_{x\to\infty} \left(2\dfrac{x+1}{x+1}\right)^2$

ANSWER: 4

(b) $\lim\limits_{x\to-\infty} \dfrac{x^2+1}{x+1}$

ANSWER: $-\infty$

9. **(10pts)** If $y^3 + y^2x = 3$, find the value of $\dfrac{dy}{dx}$ at the point $(2, 1)$.

Differentiating

$$y^3 + y^2x = 3,$$

gives that

$$3y^2y' + 2yy'x + y^2 = 0 \implies 3yy' + 2y'x + y = 0.$$

Then

$$(3y + 2x)y' = -y,$$

and

$$y' = \frac{-y}{3y + 2x}.$$

ANSWER: $-\dfrac{1}{7}$

10. **(10pts)** Let $f(x) = x^{\frac{2}{3}}$.

(a) Find the linear approximation for $f(x)$ at 27 (i.e.: an approximation valid for x near 27).

$$f(x) \approx f(27) + f'(27)(x - 27) \quad \text{for } x \text{ near } 27$$

Recall that $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ and that $27^{\frac{1}{3}} = 3$. Therefore, $f(27) = (27)^{\frac{2}{3}} = 3^2 = 9$ and $f'(27) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}$. With this, the above becomes:

$$f(x) \approx 9 + \frac{2}{9}(x - 27).$$

ANSWER: $f(x) \approx 9 + \dfrac{2}{9}(x - 27)$

(b) Use the above to calculate $(27.003)^{\frac{2}{3}}$. Calculate your answer to 5 decimal places.

$$f(27.003) \approx 9 + \frac{2}{9}(27.003 - 27) = 9 + \frac{2}{9}(0.003) = 9 + \frac{2}{3}(0.001)$$

After doing long division (by 3), we have that

$$f(27.003) \approx 9 + 2(0.000333) = 9 + 0.000666 = 9.00067.$$

ANSWER: 9.00067

11. **(10pts)** Air is pumped into a spherical balloon at a rate of $10\,\text{cm}^3/\text{min}$ (recall that the volume and the surface area of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$, respectively).

(a) What is the rate of change of the radius (in cm/min) at a moment when $r = 9\,\text{cm}$?

$$V' = 4\pi r^2 r' \implies r' = \frac{V'}{4\pi r^2}$$

ANSWER: $\frac{10}{324\pi}$

(b) What is the rate of change of the area (in cm^2/min) at the same time?

$$A' = 8\pi r r'$$

ANSWER: $8\pi 9 \frac{10}{4\pi(9)^2} = \frac{20}{9}$

12. **(10pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

(a) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

By using L'Hospital's rule twice:

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}.$$

ANSWER: $-\frac{1}{2}$

(b) $\lim_{x\to\infty} \sqrt{1 + \frac{1}{e^x}}$

$$\lim_{x\to\infty} \sqrt{1 + \frac{1}{e^x}} = \sqrt{1 + \lim_{x\to\infty} \frac{1}{e^x}} = \sqrt{1+0} = 1$$

ANSWER: 1

13. **(30pts)** Let $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x$.

(a) Find all critical numbers of f.

Note that

$$f'(x) = x^2 - 3x + 2 = (x-1)(x-2).$$

Since f is differentiable everywhere, all the critical numbers are found by solving the equation $f'(x) = 0$.

ANSWER: $1, 2$

(b) Find all intervals on which f is increasing.

Since f' is positive on $(-\infty, 1)$ and on $(2, \infty)$, f is increasing on these intervals.

ANSWER: $(-\infty, 1)$, $(2, \infty)$

(c) Find all intervals on which f is decreasing.

Since f' is negative on $(1, 2)$, f is decreasing on this interval.

ANSWER: $(1, 2)$

(d) Find all intervals on which f is concave up.

Note that

$$f''(x) = 2x - 3.$$

Because f'' is zero only at $x = \frac{3}{2}$, there are only two intervals to consider. Since f'' is positive on $(\frac{3}{2}, \infty)$, f is concave up on this interval.

ANSWER: $(\frac{3}{2}, \infty)$

(e) Find all intervals on which f is concave down.

Since f'' is negative on $(-\infty, \frac{3}{2})$, f is concave down on this interval.

ANSWER: $(-\infty, \frac{3}{2})$

(f) Find all inflection points of f.

From the above two parts, it is clear that the graph of f switches concavity at $x = \frac{3}{2}$.

ANSWER: $\frac{3}{2}$