MATH 141

MIDTERM EXAM I ANSWER KEY

October 5, 2000

1. **(20pts)**

   (a) Find the slope of the line through the points $(-1, 0)$ and $(5, 6)$.

$$m = \frac{\Delta y}{\Delta x} = \frac{6-0}{5-(-1)} = \frac{6}{6} = 1$$

ANSWER: 1

   (b) Write the equation of this line.

$$\begin{aligned} & y - 0 = 1(x - (-1)) \\ \Longrightarrow\ & y = 1(x+1) \\ \Longrightarrow\ & y = x + 1 \end{aligned}$$

ANSWER: $y = x + 1$

   (c) Find the equation of the line parallel to the above line passing through the point $(-2, 0)$.

$$\begin{aligned} & y - 0 = 1(x - (-2)) \\ \Longrightarrow\ & y = 1(x+2) \\ \Longrightarrow\ & y = x + 2 \end{aligned}$$

ANSWER: $y = x + 2$

(d) Find the equation of the line perpendicular to the above line passing through the point $(1, 0)$.

The slope of the perpendicular line is given by:

$$m' = -\frac{1}{m} = -\frac{1}{1} = -1$$

so that the line is given by:

$$y - 0 = -1(x - 1)$$
$$\implies y = -x + 1$$

ANSWER: $y = -x + 1$

2. **(20pts)**

(a) Suppose you know that $\tan(\theta) = 5$. What is $\cot(\theta)$?

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{5}$$

ANSWER: $\frac{1}{5}$

(b) What is $\sec(\theta)$ knowing that $0 < \theta < \frac{\pi}{2}$?

We know that:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\implies sec(\theta) = \pm\sqrt{1 + \tan^2(\theta)}$$
$$\implies sec(\theta) = \pm\sqrt{1 + 25} = \pm\sqrt{26}$$

Since $\cos(\theta)$, and consequently $\sec(\theta)$, is positive if $0 < \theta < \frac{\pi}{2}$, we have that

$$\sec(\theta) = \sqrt{26}$$

ANSWER: $\sqrt{26}$

(c) What is $\cos(\theta)$ with θ as above?

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \frac{1}{\sqrt{26}}$$

ANSWER: $\frac{1}{\sqrt{26}}$

(d) What is $\sin(\theta)$?

Recall:

$$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)}$$
$$\implies \sin(\theta) = \pm\sqrt{1-\frac{1}{26}}$$
$$\implies \sin(\theta) = \pm\sqrt{\frac{25}{26}} = \pm\frac{5}{\sqrt{26}}$$

Because $0 < \theta < \frac{\pi}{2}$, $\sin(\theta) > 0$ and we must take the positive value.

ANSWER: $\frac{5}{\sqrt{26}}$

3. **(10pts)** State the domain of the following functions:

(a) $\sqrt{1-x^2}$

It is necessary to have:

$$1 - x^2 > 0 \implies 1 > x^2 \implies \sqrt{x^2} < 1 \implies |x| < 1$$

ANSWER: $\{x \in \mathbb{R} \mid -1 < x < 1\}$

(b) $\arctan(\frac{1}{x})$

The range of $\arctan(y)$ is all of $\mathbb{R}$, so it is sufficient to have $x \neq 0$.

ANSWER: $\{x \in \mathbb{R} \mid x \neq 0\}$

4. **(15pts)** Solve for x in each of the following:

(a) $\ln(x) + \ln(x^3) - \ln(2x) = 3$

Using the rules for logarithms, we immediately write:

$$\ln\left(\frac{x.x^3}{2x}\right) = \ln(e^3)$$

Since ln is a one to one function, the above implies that:

$$\frac{x^3}{2} = e^3 \implies x = 2^{\frac{1}{3}}e$$

ANSWER: $2^{\frac{1}{3}}e$

(b) $e^{2x} - 2e^x + 1 = 0$

Let $y = e^x$, then the above equation becomes (note: $e^{2x} = (e^x)^2$):

$$y^2 - 2y + 1 = 0$$
$$\implies (y-1)^2 = 0 \implies y = 1$$

Thus $e^x = 1$, and we must have that $x = 0$.

ANSWER: 0

(c) $\ln(2^x) = \ln(5)$

Using the appropriate rule for logarithms, it is immediate that:

$$x \ln(2) = \ln(5) \implies x = \frac{\ln(5)}{\ln(2)}$$

ANSWER: $x = \frac{\ln(5)}{\ln(2)}$

5. **(20pts)** Evaluate the following limits (note: some of them may be $+\infty$, $-\infty$, or may not even exist):

(a)

$$\lim_{x\to 4}(x-3)^{10} = \left(\lim_{x\to 4}(x-3)\right)^{10} = 1^{10} = 1$$

ANSWER: 1

(b)

$$\lim_{x\to 1}\frac{1-2x^2}{x^2+x-2} = \lim_{x\to 1}\frac{1-2x^2}{(x-1)(x+2)}$$

Because the numerator tends to a nonzero negative number, the limit as $x \to 1^+$ is $-\infty$, while the limit as $x \to 1^-$ is $+\infty$. Therefore, the limit as $x \to 1$ does not exist.

ANSWER: Does Not Exist

(c)

$$\lim_{x\to -1}\frac{x^2-2x-3}{x^2-x-2} = \lim_{x\to -1}\frac{(x+1)(x-3)}{(x+1)(x-2)} = \lim_{x\to -1}\frac{x-3}{x-2} = \frac{-1-3}{-1-2} = \frac{4}{3}$$

ANSWER: $\frac{4}{3}$

(d)

$$\lim_{x\to 1}\sqrt{\frac{x-2}{x-5}} = \sqrt{\lim_{x\to 1}\frac{x-2}{x-5}} = \sqrt{\frac{1-2}{1-5}} = \sqrt{\frac{-1}{-4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

This is legitimate since the limit inside of the radical was positive!

ANSWER: $\frac{1}{2}$

6. **(15pts)** Let f be a function defined as follows:

$$f(x) = \begin{cases} \dfrac{1}{1+\sin(x)} & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \dfrac{2}{3-\cos(x)} & \text{if } x > 0 \end{cases}$$

(a) Evaluate the limit $\lim_{x\to 0^+} f(x)$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{2}{3-\cos(x)} = \frac{2}{\lim_{x\to 0^+}(3-\cos(x))} = \frac{2}{3-\cos(0)} = \frac{2}{3-1} = 1$$

The third equality follows from the fact that cos is a continuous function.

ANSWER: 1

(b) Evaluate the limit $\lim_{x\to 0^-} f(x)$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{1}{1+\sin(x)} = \frac{1}{\lim_{x\to 0^-}(1+\sin(x))} = \frac{1}{1+\sin(0)} = \frac{1}{1-0} = 1$$

The third equality follows from the fact that sin is a continuous function.

ANSWER: 1

(c) State if the limit $\lim_{x\to 0} f(x)$ exists. If it does exist, evaluate it.

Because the above limits agree, the limit exists and equals the same number.

ANSWER: 1

7. **(10pts)** Evaluate the following limits at ∞:

(a)

$$\lim_{x\to\infty} \frac{4x+1}{8x-\sin(x)} = \lim_{x\to\infty} \frac{4x+1}{8x-\sin(x)} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{4+\frac{1}{x}}{8-\frac{\sin(x)}{x}}$$

Using the quotient rule for limits, which it turns out may be applied, this becomes:

$$\frac{\lim\limits_{x\to\infty}\left(4+\dfrac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(8-\dfrac{\sin(x)}{x}\right)} = \frac{4}{8} = \frac{1}{2}$$

Note that $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ since $|\sin(x)| < 1$.

ANSWER: $\frac{1}{2}$

(b)

$$\lim_{x\to\infty} \sqrt{\frac{4x+2}{8x-4}} = \sqrt{\lim_{x\to\infty} \frac{4x+2}{8x-4}}$$

The limit within the radical is evaluated as follows:

$$\lim_{x\to\infty} \frac{4x+2}{8x-4} = \lim_{x\to\infty} \frac{4x+2}{8x-4} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{4+\frac{2}{x}}{8-\frac{4}{x}}$$

Applying the quotient rule gives:

$$\lim_{x\to\infty} \frac{4+\frac{2}{x}}{8-\frac{4}{x}} = \frac{\lim\limits_{x\to\infty}\left(4+\dfrac{2}{x}\right)}{\lim\limits_{x\to\infty}\left(8-\dfrac{4}{x}\right)} = \frac{4}{8} = \frac{1}{2}$$

Thus, the limit is:

$$\lim_{x\to\infty} \sqrt{\frac{4x+2}{8x-4}} = \sqrt{\frac{1}{2}}$$

ANSWER: $\sqrt{\frac{1}{2}}$

8. **(10pts)** Let f be a function defined as follows:

$$f(x) = \begin{cases} (x-2)^2 & \text{if } x < -1 \\ 1 & \text{if } x = -1 \\ \dfrac{x^2-1}{x+1} & \text{if } -1 < x < 2 \\ -1 & \text{if } x = 2 \\ -x+1 & \text{if } x > 2 \end{cases}$$

At which points is this function discontinuous?

The only values of x at which f might have a discontinuity are -1 and 2. Thus, we need to find out whether

$$\lim_{x \to -1^-} f(x) = f(-1) = \lim_{x \to -1^+} f(x)$$

and

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x).$$

Because

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x^2-1}{x+1} = \lim_{x \to -1^+} \frac{(x-1)(x+1)}{x+1} = \lim_{x \to -1^+} (x-1) = -2$$

is different from $f(-1)$ which is 1, we see that f is not continuous at $x = -1$. Also, because

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-1}{x+1} = \lim_{x \to 2^-} \frac{(x-1)(x+1)}{x+1} = \lim_{x \to 2^-} (x-1) = 1$$

is different from $f(2)$ which is -1, f is also not continuous at 2.

ANSWER: -1 and 2