MATH 141
Final Exam - Answer Key
May 7, 2001
Part I

1. **(8 points)**

   (a) Solve the inequality $17x + 3 < 14x - 2$

   $3x < -5$

   $x < -\frac{5}{3}$

   (b) Write the equation of the line parallel to the line $4x - 6y = 3$ that goes through $(1, 1)$.

   Firs find the slope of the given line:

   $6y = 4x - 3$

   $y = \frac{2}{3}x - \frac{1}{2}$

   slope $= \frac{2}{3}$

   Parallel lines have equal slopes, so we have

   $y - 1 = \frac{2}{3}(x - 1)$

   $y = \frac{2}{3}x + \frac{1}{3}$

2. **(10 points)** Find the exact value of:

   (a) $\tan(\pi) = 0$

   (b) $\sin^2(3) + \cos^2(3) = 1$

   (c) $\csc\left(\frac{\pi}{2}\right) = \dfrac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

   (d) $\frac{1}{2}\ln 9 + \ln 5 - \ln 15 = \ln 9^{\frac{1}{2}} + \ln 5 - \ln 15 = \ln \frac{3 \cdot 5}{15} = \ln 1 = 0$

   (e) $2^{\log_2 3 + \log_2 7} = 2^{\log_2 3 \cdot 7} = 2^{\log_2 21} = 21$

3. **(10 points)**

   (a) Let $f(x) = \arcsin(2x)$ and $g(x) = x^3$. Compute $h(x) = (f \circ g)(x)$

   $h(x) = f(g(x)) = f(x^3) = \arcsin(2x^3)$

(b) Find the inverse of h.

$y = \arcsin(2x^3)$

$\sin y = 2x^3$

$\frac{\sin y}{2} = x^3$

$\sqrt[3]{\frac{\sin y}{2}} = x$

$f^{-1}(y) = \sqrt[3]{\frac{\sin y}{2}}$

$f^{-1}(x) = \sqrt[3]{\frac{\sin x}{2}}$

4. **(10 points)** Solve the equations:

(a) $\ln x - \ln x^3 = -4$

$\ln x - 3\ln x = -4$

$-2\ln x = -4$

$\ln x = 2$

$x = e^2$

(b) $5^{x^2-4} = 125$

$\log_5 5^{x^2-4} = \log_5 125$

$x^2 - 4 = 3$

$x^2 = 7$

$x = \pm\sqrt{7}$

5. **(8 points)** Find c such that the function f is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} x^2 - c & , x \leq 5 \\ cx + 6 & , x > 5 \end{cases}$$

Since $x^2 - c$ and $cx + 6$ are continous everywhere, f is continuous at all points except possibly at 5. It is countinous at 5 if and only if the functions $x^2 - c$ and $cx + 6$ agree at 5, i.e.

$5^2 - c = c \cdot 5 + 6$

$6c = 19$

$c = \frac{19}{6}$

6. **(24 points)** Compute the limits (do not use L'Hospital's Rule):

(a) $\lim\limits_{x\to 3} \dfrac{x^2-7x+12}{x^2-9} = \lim\limits_{x\to 3} \dfrac{(x-4)(x-3)}{(x+3)(x-3)} = \lim\limits_{x\to 3} \dfrac{x-4}{x+3} = \dfrac{-1}{6}$

(b) $\lim\limits_{x\to 4^+} \dfrac{x^2+3x}{(x-4)(x+7)} = \left(\dfrac{\text{pos.}}{(\text{small pos.})(\text{pos.})}\right) = +\infty$

(c) $\lim\limits_{x\to 5} \dfrac{\frac{1}{x}-\frac{1}{5}}{x-5} = \lim\limits_{x\to 5} \dfrac{\frac{5-x}{5x}}{x-5} = \lim\limits_{x\to 5} \dfrac{5-x}{5x(x-5)} = \lim\limits_{x\to 5} \dfrac{-1}{5x} = -\dfrac{1}{25}$

(d) $\lim\limits_{x\to\infty} \dfrac{6x^3-3x^2+4}{x^3+7x-5} = \lim\limits_{x\to\infty} \dfrac{6-\frac{3}{x}+\frac{4}{x^3}}{1+\frac{7}{x^2}-\frac{5}{x^3}} = \dfrac{6}{1} = 6$

(e) $\lim\limits_{x\to-\infty} \dfrac{x^7+10}{x^4+3} = \lim\limits_{x\to-\infty} \dfrac{x^3+\frac{10}{x^4}}{1+\frac{3}{x^4}} = \dfrac{(-\infty)^3}{1} = -\infty$

(f) $\lim\limits_{x\to 0} x^4 \sin\left(\dfrac{1}{2x}\right) = 0$ by the Squeeze theorem because $-x^4 \le x^4 \sin\left(\frac{1}{2x}\right) \le x^4$ and $\lim_{x\to 0}(-x^4) = \lim_{x\to 0}(x^4) = 0$

(g) $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{9\sin(6x)} = \lim\limits_{x\to 0} \dfrac{\frac{\sin(4x)}{x}}{9\frac{\sin(6x)}{x}} = \lim\limits_{x\to 0} \dfrac{4\cdot\frac{\sin(4x)}{4x}}{9\cdot 6\cdot\frac{\sin(6x)}{6x}} = \dfrac{4\cdot 1}{9\cdot 6\cdot 1} = \dfrac{2}{27}$

(h) $\lim\limits_{x\to 0} \dfrac{\cos x-1}{e^x\tan(2x)} = \lim\limits_{x\to 0} \dfrac{(\cos x-1)\cos(2x)}{e^x\sin(2x)} = \lim\limits_{x\to 0} \dfrac{\frac{\cos x-1}{x}\cos(2x)}{e^x 2\frac{\sin(2x)}{2x}} = \dfrac{0\cdot 1}{1\cdot 2\cdot 1} = 0$

7. **(12 points)** Let $f(x) = \dfrac{x}{x-3}$

(a) Find the vertical and horizontal asymptotes of the graph of f.

f is not defined at 3, and $\lim\limits_{x\to 3^+} f(x) = \infty$. Therefore, $x = 3$ is a vertical asymptote.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1}{1-\frac{3}{x}} = 1$ and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{1}{1-\frac{3}{x}} = 1$, so $y = 1$ is a horizontal asymptote.

(b) Find the derivative of f using the definition.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h-3}-\frac{x}{x-3}}{h} = \lim_{h\to 0} \frac{\frac{(x+h)(x-3)-x(x+h-3)}{(x+h-3)(x-3)}}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h)(x-3)-x(x+h-3)}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{x^2+hx-3x-3h-x^2-xh+3x}{h(x+h-3)(x-3)} =$$

$$\lim_{h\to 0} \frac{-3h}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-3}{(x+h-3)(x-3)} = \frac{-3}{(x-3)^2}$$

(c) Find the tangent line to the graph of f at $(4, 4)$.

slope $= f'(4) = -3$

$y - 4 = -3(x - 4)$

$y = -3x + 16$

8. **(12 points)** Find the derivatives of:

(a) $f(x) = e^{4x} \tan 2x - \sqrt{x} \cos x^2$

$f'(x) = 4e^{4x} \tan(2x) + 2e^{4x} \sec^2(2x) - \left(\frac{1}{2\sqrt{x}} \cos(x^2) - \sqrt{x} \sin(x^2) \cdot 2x \right)$

(b) $g(x) = \dfrac{2 \cot x - x^6}{x^3 + 5}$

$g'(x) = \frac{(-2 \csc^2 x - 6x^5)(x^3+5) - (2 \cot x - x^6)(3x^2)}{(x^3+5)^2}$

(c) $h(x) = 4^{\cos(5x)} + (\cos(5x))^4$

$h'(x) = \ln 4 \cdot 4^{\cos(5x)}(-\sin(5x) \cdot 5 + 4(\cos(5x))^3(-sin(5x)) \cdot 5$

(d) $k(x) = \sqrt[3]{\sin\left(e^{-x}\right)}$

$k'(x) = \frac{1}{3}(\sin(e^{-x}))^{-\frac{2}{3}} \cos(e^{-x}) \cdot e^{-x} \cdot (-1)$

9. **(6 points)** The cost function of producing x units of some commodity is $C(x) = 1000 + 23x + 0.002x^3$. What is the marginal cost at the production level of 400 units?

Marginal cost function: $C'(x) = 23 + 0.006x^2$

$C'(400) = 23 + 0.006 \cdot 400^2 = 983$

Part II

1. **(9 points)** Differentiate the following functions:

(a) $f(x) = \arcsin(2x)$

$f'(x) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

(b) $g(x) = e^{\arctan x}$

$g'(x) = e^{\arctan x} \cdot \frac{1}{1+x^2}$

(c) $h(x) = \log_3(-\sin x)$

$h'(x) = \frac{1}{(\ln 3)(-\sin x)} \cdot (-\cos x)$

2. **(12 points)** Let $f(x) = x^3 \ln x$. Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2$

$f''(x) = 6x \ln x + 3x^2 \frac{1}{x} + 2x = 6x \ln x + 5x$

$f'''(x) = 6 \ln x + 6x \frac{1}{x} + 5 = 6 \ln x + 11$

$f^{(4)}(x) = \frac{6}{x}$

3. **(10 points)** If $xy^3 - x^2y^2 + 2y = -8$ and $y(3) = 2$, find $y'(3)$.

$y^3 + x3y^2y' - (2xy^2 + x^2 2yy') + 2y' = 0$

$y^3 + 3xy^2y' - 2xy^2 - 2x^2yy' + 2y' = 0$

$3xy^2y' - 2x^2yy' + 2y' = 2xy^2 - y^3$

$(3xy^2 - 2x^2y + 2)y' = 2xy^2 - y^3$

$y' = \frac{2xy^2 - y^3}{3xy^2 - 2x^2y + 2}$

If $x = 3$ and $y = 2$,

$y'(3) = \frac{2 \cdot 3 \cdot 4 - 8}{3 \cdot 3 \cdot 4 - 2 \cdot 9 \cdot 2 + 2} = \frac{16}{2} = 8$

4. **(10 points)** Let $f(x) = 5(x^2 + 1)^3 (\cos x)^{4x}$. Use logarithmic differentiation to find $f'(x)$.

$\ln[f(x)] = \ln[5(x^2 + 1)^3 (\cos x)^{4x}]$

$\ln[f(x)] = \ln 5 + \ln(x^2 + 1)^3 + \ln(\cos x)^{4x}$

$\ln[f(x)] = \ln 5 + 3 \ln(x^2 + 1) + 4x \ln(\cos x)$

$\frac{1}{f(x)} \cdot f'(x) = 0 + \frac{3}{x^2+1} \cdot 2x + 4 \ln(\cos x) + 4x \frac{1}{\cos x}(-\sin x)$

$f'(x) = f(x) \left(\frac{6x}{x^2+1} + 4 \ln(\cos x) - \frac{4x \sin x}{\cos x} \right)$

$f'(x) = 5(x^2 + 1)^3 (\cos x)^{4x} \left(\frac{6x}{x^2+1} + 4 \ln(\cos x) - \frac{4x \sin x}{\cos x} \right)$

5. **(10points)** Find the linearization of $f(x) = x^{3/2}$ at $x = 4$ and use it to approximate $(4.02)^{3/2}$.

$L(x) = f(x) + f'(4)(x - 4)$

$f(4) = 4^{3/2} = 8$

$f'(x) = \frac{3}{2}x^{1/2}$

$f'(4) = \frac{3}{2}4^{1/2} = 3$

$L(x) = 8 + 3(x-4) = 3x - 4$

$(4.02)^{3/2} = f(4.02) \approx L(4.02) = 3 \cdot 4.02 - 4 = 8. - 6$

6. **(12 points)** Car A starts moving north at 0.5 km/min from a point P. At the same time car B starts moving west at 1 km/min from a point 10 km due east of P. At what rate is the distance between the cars changing 6 minutes later? Is the distance increasing or decreasing at this instant?

   Let x be the distance from car A to point P, $y(x)$ the distance from car B to point P, and $z(t)$ the distance between the cars. Then $(x(t))^2 + (y(t))^2 = (z(t))^2$ (because ABP is a right triangle). Differentiate this equation with respect to time t:

   $2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t)$

   After 6 minutes $x = 0.5 \cdot 6 = 3$, $y = 10 - 1 \cdot 6 = 4$, and $z = \sqrt{x^2 + y^2} = 5$.

   Since car A goes away from point P at a speed of 0.5 km/min, $x' = 0.5$. Car B approaches P at a speed of 1 km/min, so $y' = -1$. So we have

   $2 \cdot 3 \cdot 0.5 + 2 \cdot 4 \cdot (-1) = 2 \cdot 5 \cdot z'$

   $3 - 8 = 10z'$

   $z' = -\frac{1}{2}$ (km/min)

7. **(15 points)** Let $f(x) = x^4 + 2x^3 - 5$.

   (a) Find the critical numbers of $f(x)$.

   $f'(x) = 4x^3 + 6x^2 = 2x^2(2x+3) = 0$ when $x = 0$ or $x = -\frac{3}{2}$ $\Rightarrow$ 0 and $-\frac{3}{2}$ are critical numbers.

   (b) Where is $f(x)$ increasing? Decreasing?

   $f'(x) > 0$ on $(-\frac{3}{2}, +\infty)$ $\Rightarrow$ $f(x)$ is increasing on $(-\frac{3}{2}, +\infty)$. $f'(x) < 0$ on $(-\infty, -\frac{3}{2})$ $\Rightarrow$ $f(x)$ is decreasing on $(-\infty, -\frac{3}{2})$.

   (c) Find local maxima and minima of $f(x)$.

$f'(x)$ changes from negative to positive at $-\frac{3}{2}$ $\Rightarrow$ $f(x)$ has a local minimum at $-\frac{3}{2}$.

(d) Where is $f(x)$ concave upward? Concave downward?

$f''(x) = 12x^{12} + 12x = 12x(x+1)$

$f''(x) > 0$ on $(-\infty, -1)$ and $(0, +\infty)$ $\Rightarrow$ $f(x)$ is concave upward on $(-\infty, -1)$ and $(0, +\infty)$.

$f''(x) < 0$ on$(-1, 0)$ $\Rightarrow$ $f(x)$ is concave downward on $(-1, 0)$.

(e) Find the inflection points of $f(x)$.

$f(x)$ changes the direction of concavity at -1 and 0 $\Rightarrow$ -1 and 0 are inflection points.

8. **(10 points)** Show that the equation $x^5 + 3x^3 + 5x + 7 = 0$ has exactly one root in the interval $[-1, 1]$.

Let $f(x) = x^5 + 3x^3 + 5x + 7$.

Since $f(-1) = -2 < 0$ and $f(1) = 16 > 0$, by the Intermediate Value Theorem $f(x)$ has at least one root between -1 and 1.

Suppose $f(x)$ has two roots. Then by Rolle's Theorem there exists a point c s.t. $f'(c) = 0$. But $f'(x) = 5x^4 + 9x^2 + 5 > 0$ for all x. Therefore, $f(x)$ has exactly one root.

9. **(12 points)** Evaluate the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\tan x}{e^x - 1} \left(= \dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{(\tan x)'}{(e^x-1)'} = \lim\limits_{x\to 0} \dfrac{(\sec x)^2}{e^x} = \dfrac{1}{1} = 1$

(b) $\lim\limits_{x\to\infty} x^2 e^{-2x} = \lim\limits_{x\to\infty} \dfrac{x^2}{e^{2x}} \left(= \dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty} \dfrac{(x^2)'}{(e^{2x})'} = \lim\limits_{x\to\infty} \dfrac{2x}{2e^{2x}} \left(= \dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty} \dfrac{(2x)'}{(2e^{2x})'} =$
$\lim\limits_{x\to\infty} \dfrac{2}{4e^{2x}} = \dfrac{2}{\infty} = 0$

(c) $\lim\limits_{x\to 0^+} (-\ln x)^x = \lim\limits_{x\to 0^+} \left(e^{-\ln x}\right)^x = \lim\limits_{x\to 0^+} e^{-(\ln x)x} = e^{\lim\limits_{x\to 0^+}(-(\ln x)x)} = e^{-\lim\limits_{x\to 0^+} x\ln x}$

$\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x} = \lim\limits_{x\to 0^+} \dfrac{1/x}{-1/x^2} = \lim\limits_{x\to 0^+}(-x) = 0$

Thus $\lim\limits_{x\to 0^+} (-\ln x)^x = e^{-\lim\limits_{x\to 0^+} x\ln x} = e^{-0} = 1$