MATH 141
Final Exam - Answer Key
May 7, 2001
Part I

1. (8 points)

(a) Solve the inequality $17x + 3 < 14x - 2$

$3x < -5$

$x < -\frac{5}{3}$

(b) Write the equation of the line parallel to the line $4x - 6y = 3$ that goes through $(1, 1)$.

First find the slope of the given line:

$6y = 4x - 3$

$y = \frac{2}{3}x - \frac{1}{2}$

slope $= \frac{2}{3}$

Parallel lines have equal slopes, so we have

$y - 1 = \frac{2}{3}(x - 1)$

$y = \frac{2}{3}x + \frac{1}{3}$

2. (10 points) Find the exact value of:

(a) $\tan(\pi) = 0$

(b) $\sin^2(3) + \cos^2(3) = 1$

(c) $\csc\left(\frac{\pi}{2}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

(d) $\frac{1}{2}\ln 9 + \ln 5 - \ln 15 = \ln 9^{\frac{1}{2}} + \ln 5 - \ln 15 = \ln \frac{5}{15} = \ln 1 = 0$

(e) $2^{\log_2 3 + \log_2 7} = 2^{\log_2 21} = 21$

3. (10 points)

(a) Let $f(x) = \arcsin(2x)$ and $g(x) = x^3$. Compute $h(x) = (f \circ g)(x)$

$h(x) = f(g(x)) = f(x^3) = \arcsin(2x^3)$
(b) Find the inverse of \( h \).

\[
y = \arcsin(2x^3)
\]
\[
\sin y = 2x^3
\]
\[
\sin \frac{y}{2} = x^3
\]
\[
\sqrt[3]{\sin \frac{y}{2}} = x
\]
\[
f^{-1}(y) = \sqrt[3]{\sin \frac{y}{2}}
\]
\[
f^{-1}(x) = \sqrt[3]{{\sin x \over 2}}
\]

4. (10 points) Solve the equations:

(a) \( \ln x - \ln x^3 = -4 \)

\[
\ln x - 3 \ln x = -4
\]
\[
-2 \ln x = -4
\]
\[
\ln x = 2
\]
\[
x = e^2
\]

(b) \( 5^{x^2-4} = 125 \)

\[
\log_5 5^{x^2-4} = \log_5 125
\]
\[
x^2 - 4 = 3
\]
\[
x^2 = 7
\]
\[
x = \pm \sqrt{7}
\]

5. (8 points) Find \( c \) such that the function \( f \) is continuous on \( \mathbb{R} \).

\[
f(x) = \begin{cases} 
  x^2 - c, & x \leq 5 \\
  cx + 6, & x > 5 
\end{cases}
\]

Since \( x^2 - c \) and \( cx + 6 \) are continuous everywhere, \( f \) is continuous at all points except possibly at 5. It is continuous at 5 if and only if the functions \( x^2 - c \) and \( cx + 6 \) agree at 5, i.e.

\[
5^2 - c = c \cdot 5 + 6
\]
\[
6c = 19
\]
c = \frac{19}{6}

6. (24 points) Compute the limits (do not use L’Hospital’s Rule):
   
   (a) \lim_{x \to 3} \frac{x^2 - 7x + 12}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 4)(x - 3)}{(x + 3)(x - 3)} = \lim_{x \to 3} \frac{x - 4}{x + 3} = -\frac{1}{6}

   (b) \lim_{x \to 4^+} \frac{x^2 + 3x}{(x - 4)(x + 7)} = \left( \frac{\text{pos.}}{\text{small pos.}(\text{pos.})} \right) = +\infty

   (c) \lim_{x \to 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} = \lim_{x \to 5} \frac{5 - x}{5x(x - 5)} = \lim_{x \to 5} \frac{-1}{5x} = -\frac{1}{25}

   (d) \lim_{x \to \infty} \frac{6x^3 - 3x^2 + 4}{x^3 + 7x - 5} = \lim_{x \to \infty} \frac{6 - \frac{3}{x} + \frac{4}{x^3}}{1 + \frac{7}{x} - \frac{5}{x^3}} = \frac{6}{1} = 6

   (e) \lim_{x \to -\infty} \frac{x^7 + 10}{x^4 + 3} = \lim_{x \to -\infty} \frac{x^3 + \frac{10}{x^4}}{1 + \frac{3}{x^4}} = (\infty)^3 = -\infty

   (f) \lim_{x \to 0} x^4 \sin \left( \frac{1}{2x} \right) = 0 \text{ by the Squeeze theorem because } -x^4 \leq x^4 \sin \left( \frac{1}{2x} \right) \leq x^4 \text{ and } \lim_{x \to 0} (-x^4) = \lim_{x \to 0} (x^4) = 0

   (g) \lim_{x \to 0} \frac{\sin(4x)}{9 \sin(6x)} = \lim_{x \to 0} \frac{\sin(4x)}{9 \cdot 6 \cdot \sin(6x) / 6x} = \lim_{x \to 0} \frac{4 \cdot \sin(4x) / 4x}{9 \cdot 6 \cdot 1 / 6} = \frac{2}{27}

   (h) \lim_{x \to 0} e^x \tan(2x) = \lim_{x \to 0} \frac{(\cos x - 1) \cos(2x)}{e^x \sin(2x)} = \lim_{x \to 0} \frac{\cos x - 1 \cos(2x)}{e^x \sin(2x) / 2x} = \frac{0 \cdot 1}{1 \cdot 2 \cdot 1} = 0

7. (12 points) Let \( f(x) = \frac{x}{x - 3} \)

   (a) Find the vertical and horizontal asymptotes of the graph of \( f \).

   \( f \) is not defined at 3, and \( \lim_{x \to 3^+} f(x) = \infty \). Therefore, \( x = 3 \) is a vertical asymptote.

   \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1 - \frac{3}{x}} = 1 \) and \( \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{1 - \frac{3}{x}} = 1 \), so \( y = 1 \) is a horizontal asymptote.

   (b) Find the derivative of \( f \) using the definition.

   \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{x + h - 3}{h(x + h + 3)(x - 3)} = \lim_{h \to 0} \frac{x^2 + hx - 3x - 3h - x^2 - xh + 3x}{h(x + h + 3)(x - 3)} = \lim_{h \to 0} \frac{-3h}{h(x + h + 3)(x - 3)} = -\frac{3}{(x - 3)^2} \]
(c) Find the tangent line to the graph of \( f \) at \((4, 4)\).

\[
\text{slope} = f'(4) = -3
\]
\[
y - 4 = -3(x - 4)
\]
\[
y = -3x + 16
\]

8. (12 points) Find the derivatives of:

(a) \( f(x) = e^{4x} \tan 2x - \sqrt{x} \cos x^2 \)

\[
f'(x) = 4e^{4x} \tan(2x) + 2e^{4x} \sec^2(2x) - \left( \frac{1}{2\sqrt{x}} \cos(x^2) - \sqrt{x} \sin(x^2) \cdot 2x \right)
\]

(b) \( g(x) = \frac{2 \cot x - x^6}{x^3 + 5} \)

\[
g'(x) = \frac{(-2\cot^2 x - 6x^5)(x^3 + 5) - (2\cot x - x^6)(3x^2)}{(x^3 + 5)^2}
\]

(c) \( h(x) = 4^{\cos(5x)} + (\cos(5x))^4 \)

\[
h'(x) = \ln 4 \cdot 4^{\cos(5x)}(-\sin(5x)) \cdot 5 + 4(\cos(5x))^3(-\sin(5x)) \cdot 5
\]

(d) \( k(x) = \sqrt[3]{\sin(e^{-x})} \)

\[
k'(x) = \frac{1}{3}(\sin(e^{-x}))^{-\frac{2}{3}} \cos(e^{-x}) \cdot e^{-x} \cdot (-1)
\]

9. (6 points) The cost function of producing \( x \) units of some commodity is \( C(x) = 1000 + 23x + 0.002x^3 \). What is the marginal cost at the production level of 400 units?

Marginal cost function: \( C'(x) = 23 + 0.006x^2 \)

\[
C'(400) = 23 + 0.006 \cdot 400^2 = 983
\]

**Part II**

1. (9 points) Differentiate the following functions:

(a) \( f(x) = \arcsin(2x) \)

\[
f'(x) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2
\]

(b) \( g(x) = e^{\arctan x} \)

\[
g'(x) = e^{\arctan x} \cdot \frac{1}{1+x^2}
\]

(c) \( h(x) = \log_3 (-\sin x) \)

\[
h'(x) = \frac{1}{(\ln 3)(-\sin x)} \cdot (-\cos x)
\]
2. (12 points) Let \( f(x) = x^3 \ln x \). Find \( f'(x) \), \( f''(x) \), \( f'''(x) \), and \( f^{(4)}(x) \).

\[
\begin{align*}
f'(x) &= 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2 \\
f''(x) &= 6x \ln x + 3x^2 \frac{1}{x} + 2x = 6x \ln x + 5x \\
f'''(x) &= 6 \ln x + 6x \frac{1}{x} + 5 = 6 \ln x + 11 \\
f^{(4)}(x) &= \frac{6}{x}
\end{align*}
\]

3. (10 points) If \( xy^3 - x^2y^2 + 2y = -8 \) and \( y(3) = 2 \), find \( y'(3) \).

\[
\begin{align*}
y^3 + x3y^2y' - (2xy^2 + x^2y y') + 2y' &= 0 \\
y^3 + 3xy^2y' - 2xy^2 - 2x^2y y' + 2y' &= 0 \\
3xy^2y' - 2x^2yy' + 2y' &= 2xy^2 - y^3 \\
(3xy^2 - 2x^2y + 2)y' &= 2xy^2 - y^3 \\
y' &= \frac{2xy^2 - y^3}{3xy^2 - 2x^2y + 2}
\end{align*}
\]

If \( x = 3 \) and \( y = 2 \),

\[
y'(3) = \frac{2 \cdot 3^2 - 2^3}{3 \cdot 2^2 - 2 \cdot 2^2 + 2} = \frac{16}{2} = 8
\]

4. (10 points) Let \( f(x) = 5(x^2 + 1)^3(\cos x)^4x \). Use logarithmic differentiation to find \( f'(x) \).

\[
\begin{align*}
\ln[f(x)] &= \ln[5(x^2 + 1)^3(\cos x)^4x] \\
\ln[f(x)] &= \ln 5 + \ln(x^2 + 1)^3 + \ln(\cos x)^4x \\
\ln[f(x)] &= \ln 5 + 3 \ln(x^2 + 1) + 4x \ln(\cos x) \\
\frac{1}{f(x)} \cdot f'(x) &= 0 + \frac{3}{x^2 + 1} \cdot 2x + 4 \ln(\cos x) + 4x \frac{1}{\cos x} (-\sin x) \\
f'(x) &= f(x) \left( \frac{6x}{x^2 + 1} + 4 \ln(\cos x) - \frac{4x \sin x}{\cos x} \right) \\
f'(x) &= 5(x^2 + 1)^3(\cos x)^4x \left( \frac{6x}{x^2 + 1} + 4 \ln(\cos x) - \frac{4x \sin x}{\cos x} \right)
\end{align*}
\]

5. (10 points) Find the linearization of \( f(x) = x^{3/2} \) at \( x = 4 \) and use it to approximate \( (4.02)^{3/2} \).

\[
L(x) = f(x) + f'(4)(x - 4) \\
f(4) = 4^{3/2} = 8
\]
\[ f'(x) = \frac{3}{2} x^{1/2} \]
\[ f'(4) = \frac{3}{2} 4^{1/2} = 3 \]
\[ L(x) = 8 + 3(x - 4) = 3x - 4 \]
\[ (4.02)^{3/2} = f(4.02) \approx L(4.02) = 3 \cdot 4.02 - 4 = 8.6 \]

6. (12 points) Car \( A \) starts moving north at \( 0.5 \) \( \text{km/min} \) from a point \( P \). At the same time car \( B \) starts moving west at \( 1 \) \( \text{km/min} \) from a point \( 10 \) \( \text{km} \) due east of \( P \). At what time rate is the distance between the cars changing 6 minutes later? Is the distance increasing or decreasing at this instant?

Let \( x \) be the distance from car \( A \) to point \( P \), \( y(x) \) the distance from car \( B \) to point \( P \), and \( z(t) \) the distance between the cars. Then \( (x(t))^2 + (y(t))^2 = (z(t))^2 \) (because \( ABP \) is a right triangle). Differentiate this equation with respect to time \( t \):

\[ 2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t) \]

After 6 minutes \( x = 0.5 \cdot 6 = 3, \ y = 10 - 1 \cdot 6 = 4, \) and \( z = \sqrt{x^2 + y^2} = 5 \).

Since car \( A \) goes away from point \( P \) at a speed of \( 0.5 \) \( \text{km/min} \), \( x' = 0.5 \). Car \( B \) approaches \( P \) at a speed of \( 1 \) \( \text{km/min} \), so \( y' = -1 \). So we have

\[ 2 \cdot 3 \cdot 0.5 + 2 \cdot 4 \cdot (-1) = 2 \cdot 5 \cdot z' \]
\[ 3 - 8 = 10z' \]
\[ z' = -\frac{1}{2} \text{ (km/min)} \]

7. (15 points) Let \( f(x) = x^4 + 2x^3 - 5 \).

(a) Find the critical numbers of \( f(x) \).
\[ f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) = 0 \]
when \( x = 0 \) or \( x = -\frac{3}{2} \Rightarrow \)
0 and \(-\frac{3}{2}\) are critical numbers.

(b) Where is \( f(x) \) increasing? Decreasing?
\[ f'(x) > 0 \text{ on } (-\frac{3}{2}, +\infty) \Rightarrow f(x) \text{ is increasing on } (-\frac{3}{2}, +\infty). \]
\[ f'(x) < 0 \text{ on } (-\infty, -\frac{3}{2}) \Rightarrow f(x) \text{ is decreasing on } (-\infty, -\frac{3}{2}). \]

(c) Find local maxima and minima of \( f(x) \).
\[ f'(x) \text{ changes from negative to positive at } -\frac{3}{2} \Rightarrow f(x) \text{ has a local minimum at } -\frac{3}{2}. \]

(d) Where is \( f(x) \) concave upward? Concave downward?
\[ f''(x) = 12x^2 + 12x = 12x(x + 1) \]
\[ f''(x) > 0 \text{ on } (-\infty, -1) \text{ and } (0, +\infty) \Rightarrow f(x) \text{ is concave upward on } (-\infty, -1) \text{ and } (0, +\infty). \]
\[ f''(x) < 0 \text{ on } (-1, 0) \Rightarrow f(x) \text{ is concave downward on } (-1, 0). \]

(e) Find the inflection points of \( f(x) \).
\( f(x) \) changes the direction of concavity at \(-1\) and \(0 \Rightarrow -1 \text{ and } 0 \) are inflection points.

8. (10 points) Show that the equation \( x^5 + 3x^3 + 5x + 7 = 0 \) has exactly one root in the interval \([-1, 1]\).

Let \( f(x) = x^5 + 3x^3 + 5x + 7 \).

Since \( f(-1) = -2 < 0 \) and \( f(1) = 16 > 0 \), by the Intermediate Value Theorem \( f(x) \)
has at least one root between \(-1\) and \(1\).

Suppose \( f(x) \) has two roots. Then by Rolle’s Theorem there exists a point \( c \) s.t. \( f'(c) = 0 \). But \( f'(x) = 5x^4 + 9x^2 + 5 > 0 \) for all \( x \). Therefore, \( f(x) \) has exactly one root.

9. (12 points) Evaluate the following limits:

(a) \[ \lim_{x \to 0} \frac{\tan x}{e^x - 1} = \lim_{x \to 0} \frac{(\tan x)'}{e^x - 1}' = \lim_{x \to 0} \frac{(\sec x)^2}{e^x} = \frac{1}{1} = 1 \]

(b) \[ \lim_{x \to \infty} \frac{x^2}{e^{2x}} = \lim_{x \to \infty} \frac{x^2}{e^{2x}} = \lim_{x \to \infty} \frac{(x^2)'}{e^{2x}} = \lim_{x \to \infty} \frac{2x}{2e^{2x}} = \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{2x}{2e^{2x}} = 0 \]

(c) \[ \lim_{x \to 0^+} (-\ln x)^x = \lim_{x \to 0^+} (e^{-\ln x})^x = \lim_{x \to 0^+} e^{-\ln x} = e^{x \to 0^+} (-\ln x) = e^x = e \]
\[ \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0 \]

Thus \( \lim_{x \to 0^+} (-\ln x)^x = e \lim_{x \to 0^+} x \ln x = e^0 = 1 \)