

MATH 141
Midterm 1 - Answer key
February 22, 2001

1. **(10 points)**) Solve the inequalities:

(a) $4 < 2x + 1 < 8$

$$3 < 2x < 7$$

$$\frac{3}{2} < x < \frac{7}{2}$$

(b) $|5x - 2| < 7$

$$-7 < 5x - 2 < 7$$

$$-5 < 5x < 9$$

$$-1 < x < \frac{9}{5}$$

2. **(10 points)**

(a) Find the slope of the line through the points (2, 1) and (1, 6).

$$m = \frac{6-1}{1-2} = -5$$

(b) Write the equation of this line.

$$y - 1 = -5(x - 2)$$

$$y = -5x + 11$$

(c) Write the equation of the line parallel to the line in (b) that goes through (-1, -2).

$$\text{slope} = -5$$

$$y + 2 = -5(x + 1)$$

$$y = -5x - 7$$

(d) Write the equation of the line perpendicular to the line in (b) that goes through (5, 3).

$$\text{slope} = \frac{1}{5}$$

$$y - 3 = \frac{1}{5}(x - 5)$$

$$y = \frac{1}{5}x + 2$$

3. (10points)

(a) Convert from degrees to radians:

$$270^\circ = 270 \cdot \frac{\pi}{180} \text{ rad} = \frac{3\pi}{2} \text{ rad}$$

$$135^\circ = 135 \cdot \frac{\pi}{180} \text{ rad} = \frac{3\pi}{4} \text{ rad}$$

(b) Convert from radians to degrees:

$$\frac{\pi}{6} = \frac{180^\circ}{6} = 30^\circ$$

$$\frac{-7\pi}{2} = -\frac{7 \cdot 180^\circ}{2} = -630^\circ$$

(c) Find the exact value of:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\cos 3\pi = -1$$

$$\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$\sin(\arcsin 1) = 1$$

$$\sec(0) = 1$$

4. (10 points) Let $f(x) = \sqrt{x}$ and $g(x) = 3 - x$. Compute $f \circ g$ and $g \circ f$ and find their domains.

$$f \circ g = \sqrt{x-3} \quad \text{Domain} = (-\infty, 3]$$

$$g \circ f = 3 - \sqrt{x} \quad \text{Domain} = [0, +\infty)$$

5. (8 points) Find the inverse of the function $f(x) = \ln(\sqrt{x})$

$$y = \ln(\sqrt{x})$$

$$e^y = \sqrt{x}$$

$$(e^y)^2 = x$$

$$f^{-1}(y) = e^{2y}$$

$$f^{-1}(x) = e^{2x}$$

6. (12 points) Solve the equations:

(a) $\ln(2x - 1) = -2$

$$2x - 1 = e^{-2}$$

$$x = \frac{e^{-2} + 1}{2}$$

(b) $3^{x-1} = 4$

$$x - 1 = \log_3 4$$

$$x = \log_3 4 + 1$$

- (c) $8^{2x} = 8^{x^2+1}$
 $2x = x^2 + 1$
 $x^2 - 2x + 1 = 0$
 $(x - 1)^2 = 0$
 $x = 1$
- (d) $\ln(x) - \ln 9 = \ln(x^3)$
 $\ln\left(\frac{x}{9}\right) = \ln(x^3)$
 $\frac{x}{9} = x^3$
 $x = 9x^3$
 $9x^3 - x = 0$
 $x(3x - 1)(3x + 1) = 0$
 $x = 0, \frac{1}{3}, -\frac{1}{3}$
 Throw away solutions 0 and $-\frac{1}{3}$ because $\ln 0$ and $\ln\left(-\frac{1}{3}\right)$ are not defined.

7. (15 points) Compute the limits:

- (a) $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 5$
- (b) $\lim_{x \rightarrow 0} \frac{\sqrt{4 + x^2} - 2}{x^2} = \lim_{x \rightarrow 0} \frac{(\sqrt{4 + x^2} - 2)(\sqrt{4 + x^2} + 2)}{x^2(\sqrt{4 + x^2} + 2)} =$
 $= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{4 + x^2} + 2)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{4 + x^2} + 2} = \frac{1}{4}$
- (c) $\lim_{x \rightarrow -1} \sqrt{\frac{x^3 + 2}{x^2 + 4}} = \sqrt{\frac{(-1)^3 + 2}{(-1)^2 + 4}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$
- (d) $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{2}{x}\right) = 0$ because
 $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$
 $-x^2 \leq x^2 \cos\left(\frac{2}{x}\right) \leq x^2$
 $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$
 therefore, by the squeeze theorem, $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{2}{x}\right) = 0$
- (e) $\lim_{x \rightarrow 1^-} \frac{x^2 - 2x + 7}{(x - 1)(x + 3)} = \left(\frac{6}{(\text{small negative})(4)} \right) = -\infty$

8. (5 points) Find the discontinuities of the function $f(x) = \frac{\sin(x)}{x+1}$.

$f(x)$ is not defined at $x = -1$, therefore, it is discontinuous at $x = -1$.
It is continuous at all other points.

9. (10 points) Find c such that the function f is continuous on \mathbb{R} .

$$f(x) = \begin{cases} 2x + 1 & , x \leq 4 \\ cx - 3 & , x > 4 \end{cases}$$

Since both $2x + 1$ and $cx - 3$ are continuous on \mathbb{R} , the function f is continuous at all points except possibly at 4. So we only have to make sure that f is continuous at 4.

Solution 1. By definition, f is continuous at 4 if

$$\lim_{x \rightarrow 4^-} f(x) = f(4) = \lim_{x \rightarrow 4^+} f(x).$$

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (2x + 1) = 9,$$

$$f(4) = 9.$$

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} (cx - 3) = 4c - 3.$$

Thus f is continuous at 4, if

$$9 = 4c - 3$$

$$c = 3$$

Solution 2. f is continuous at 4 if the values of the functions $2x + 1$ and $cx - 3$ are equal at 4:

$$2 \cdot 4 + 1 = c \cdot 4 - 3$$

$$c = 3$$

10. (10 points) Show that the equation $\cos(x) - x^2 = 0$ has at least one solution in the interval $(0, 1)$.

Let $f(x) = \cos(x) - x^2$.

$$f(0) = \cos(0) - 0^2 = 1 > 0,$$

$$f(1) = \cos(1) - 1^2 < 0 \text{ because } \cos(1) < 1.$$

Therefore, by the intermediate value theorem, the equation $f(x) = 0$ has a solution on $(0, 1)$.