MATH 141
Midterm 1 - Answer key
February 22, 2001

1. **(10 points)** Solve the inequalities:

   (a) $4 < 2x + 1 < 8$
   $3 < 2x < 7$
   $\frac{3}{2} < x < \frac{7}{2}$

   (b) $|5x - 2| < 7$
   $-7 < 5x - 2 < 7$
   $-5 < 5x < 9$
   $-1 < x < \frac{9}{5}$

2. **(10 points)**

   (a) Find the slope of the line through the points $(2, 1)$ and $(1, 6)$.
   $m = \frac{6-1}{1-2} = -5$

   (b) Write the equation of this line.
   $y - 1 = -5(x - 2)$
   $y = -5x + 11$

   (c) Write the equation of the line parallel to the line in (b) that goes through $(-1, -2)$.
   slope $= -5$
   $y + 2 = -5(x + 1)$
   $y = -5x - 7$

   (d) Write the equation of the line perpendicular to the line in (b) that goes through $(5, 3)$.
   slope $= \frac{1}{5}$
   $y - 3 = \frac{1}{5}(x - 5)$
   $y = \frac{1}{5}x + 2$

3. **(10points)**

   (a) Convert from degrees to radians:
   $270° = 270 \cdot \frac{\pi}{180}$ rad $= \frac{3\pi}{2}$ rad
   $135° = 135 \cdot \frac{\pi}{180}$ rad $= \frac{3\pi}{4}$ rad

   (b) Convert from radians to degrees:
   $\frac{\pi}{6} = \frac{180°}{6} = 30°$
   $\frac{-7\pi}{2} = -\frac{7 \cdot 180°}{2} = -630°$

   (c) Find the exact value of:
   $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
   $\cos 3\pi = -1$
   $\tan(\frac{2\pi}{3}) = -\sqrt{3}$
   $\arctan(\sqrt{3}) = \frac{\pi}{3}$
   $\sin(\arcsin 1) = 1$
   $\sec(0) = 1$

4. **(10 points)** Let $f(x) = \sqrt{x}$ and $g(x) = 3 - x$. Compute $f \circ g$ and $g \circ f$ and find their domains.

   $f \circ g = \sqrt{x - 3}$ Domain $= (-\infty, 3]$
   $g \circ f = 3 - \sqrt{x}$ Domain $= [0, +\infty)$

5. **(8 points)** Find the inverse of the function $f(x) = \ln(\sqrt{x})$

   $y = \ln(\sqrt{x})$
   $e^y = \sqrt{x}$
   $(e^y)^2 = x$
   $f^{-1}(y) = e^{2y}$
   $f^{-1}(x) = e^{2x}$

6. **(12 points)** Solve the equations:

   (a) $\ln(2x - 1) = -2$
   $2x - 1 = e^{-2}$
   $x = \frac{e^{-2}+1}{2}$

   (b) $3^{x-1} = 4$
   $x - 1 = \log_3 4$
   $x = \log_3 4 + 1$

(c) $8^{2x} = 8^{x^2+1}$
$2x = x^2 + 1$
$x^2 - 2x + 1 = 0$
$(x-1)^2 = 0$
$x = 1$

(d) $\ln(x) - \ln 9 = \ln(x^3)$
$\ln\left(\frac{x}{9}\right) = \ln(x^3)$
$\frac{x}{9} = x^3$
$x = 9x^3$
$9x^3 - x = 0$
$x(3x-1)(3x+1) = 0$
$x = 0, \quad \frac{1}{3}, \quad -\frac{1}{3}$
Throw away solutions 0 and $-\frac{1}{3}$ because $\ln 0$ and $\ln\left(-\frac{1}{3}\right)$ are not defined.

7. **(15 points)** Compute the limits:

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 5$

(b) $\lim_{x\to 0} \frac{\sqrt{4+x^2}-2}{x^2} = \lim_{x\to 0} \frac{(\sqrt{4+x^2}-2)(\sqrt{4+x^2}+2)}{x^2(\sqrt{4+x^2}+2)} =$

$$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{4+x^2}+2)} = \lim_{x\to 0} \frac{1}{\sqrt{4+x^2}+2} = \frac{1}{4}$$

(c) $\lim_{x\to -1} \sqrt{\frac{x^3+2}{x^2+4}} = \sqrt{\frac{(-1)^3+2}{(-1)^2+4}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$

(d) $\lim_{x\to 0} x^2 \cos\left(\frac{2}{x}\right) = 0$ because
$-1 \le \cos\left(\frac{2}{x}\right) \le 1$
$-x^2 \le x^2\cos\left(\frac{2}{x}\right) \le x^2$
$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$
therefore, by the squeeze theorem, $\lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right) = 0$

(e) $\lim_{x\to 1^-} \frac{x^2-2x+7}{(x-1)(x+3)} = \left(\frac{6}{(small\ negative)(4)}\right) = -\infty$

8. **(5 points)** Find the discontinuities of the function $f(x) = \dfrac{\sin(x)}{x+1}$.

   $f(x)$ is not defined at $x = -1$, therefore, it is discontinuous at $x = -1$. It is continuous at all other points.

9. **(10 points)** Find c such that the function f is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} 2x+1 & ,x \leq 4 \\ cx-3 & ,x > 4 \end{cases}$$

   Since both $2x+1$ and $cx-3$ are continuous on $\mathbb{R}$, the function f is continuous at all points except possibly at 4. So we only have to make sure that f is continuous at 4.

   Solution 1. By definition, f is continuous at 4 if

   $\lim_{x\to 4^-} f(x) = f(4) = \lim_{x\to 4^+} f(x)$.
   $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (2x+1) = 9$,
   $f(4) = 9$.
   $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (cx-3) = 4c-3$.
   Thus f is continuous at 4, if
   $9 = 4c - 3$
   $c = 3$

   Solution 2. f is continuous at 4 if the values of the functions $2x+1$ and $cs-3$ are equal at 4:

   $2 \cdot +1 = c \cdot 4 - 3$
   $c = 3$

10. **(10 points)** Show that the equation $\cos(x) - x^2 = 0$ has at least one solution in the interval $(0, 1)$.

    Let $f(x) = \cos(x) - x^2$.
    $f(0) = \cos(0) - 0^2 = 1 > 0$,
    $f(1) = \cos(1) - 1^2 < 0$ because $\cos(1) < 1$.
    Therefore, by the intermediate value theorem, the equation $f(x) = 0$ has a solution on $(0, 1)$.