MATH 141
Midterm 2 - Answer key
April 5, 2001

1. (20 points) Compute the limits:

   (a) \[ \lim_\limits{x \to \infty} \frac{6x^2 + 5x}{(1 - x)(2x - 3)} = \lim_\limits{x \to \infty} \frac{6x^2 + 5x}{-2x^2 + 5x - 3} = \]
   \[ = \lim_\limits{x \to \infty} \frac{6 + \frac{5}{x}}{-2 + \frac{5}{x} - \frac{3}{x^2}} = \frac{6}{-2} = -3 \]

   (b) \[ \lim_\limits{x \to \infty} \frac{x^2 + 4x + 1}{x^5 - 3x^3 + 4} = \lim_\limits{x \to \infty} \frac{\frac{1}{x^3} + \frac{4}{x} + \frac{1}{x^5}}{1 - \frac{3}{x^2} + \frac{4}{x^5}} = 0 = 0 \]

   (c) \[ \lim_\limits{x \to \infty} \sqrt{x^2 + 2x + 2} - x = \lim_\limits{x \to \infty} \frac{(\sqrt{x^2 + 2x + 2} - x)(\sqrt{x^2 + 2x + 2} + x)}{\sqrt{x^2 + 2x + 2} + x} = \]
   \[ = \lim_\limits{x \to \infty} \frac{2x + 2}{\sqrt{x^2 + 2x + 2} + x} = \lim_\limits{x \to \infty} \frac{2 + \frac{2}{x}}{\sqrt{1 + \frac{2}{x} + \frac{2}{x^2}} + 1} = \frac{2}{\sqrt{1}} = 1 \]

   (d) \[ \lim_\limits{x \to \infty} (x + \sqrt{x})(x^2 + 4) = (\infty + \infty)(\infty + 4) = \infty \]

   (e) \[ \lim_\limits{x \to \infty} e^{-\frac{2}{x^2}} = e^{\left( \lim_\limits{x \to \infty} -\frac{3}{x^2} \right)} = e^0 = 1 \]

2. (10 points) Find the horizontal asymptotes of the graph of the function \( f(x) = \frac{x - 3}{\sqrt{x^2 + 3x + 2}} \)

   \[ \lim_\limits{x \to \infty} f(x) = \lim_\limits{x \to \infty} \frac{1 - \frac{3}{x}}{\sqrt{\frac{x^2 + 3x + 2}{x}}} = \lim_\limits{x \to \infty} \frac{1 - \frac{3}{x}}{\sqrt{1 + \frac{3}{x} + \frac{2}{x^2}}} = 1 \]

   \[ \lim_\limits{x \to -\infty} f(x) = \lim_\limits{x \to -\infty} \frac{1 - \frac{3}{x}}{\sqrt{\frac{x^2 + 3x + 2}{x}}} = \lim_\limits{x \to -\infty} \frac{1 - \frac{3}{x}}{-\sqrt{1 + \frac{3}{x} + \frac{2}{x^2}}} = -1 \]

   Therefore, the horizontal asymptotes are \( y = 1 \) and \( y = -1 \).
3. (10 points) Let $f(x) = x^2 - \frac{2}{x}$.

(a) Find $f'(x)$ using the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - \frac{2}{x+h} - (x^2 - \frac{2}{x})}{h} =$$

$$= \lim_{h \to 0} \frac{2x - \frac{2}{x+h} + x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2x + 2xh - 2x}{x(x+h)} + 2xh + h^2$$

$$= \lim_{h \to 0} \left( \frac{2}{x(x+h)} + 2x + h \right) = \frac{2}{x^2} + 2x$$

(b) Find the tangent to the graph of $f$ at $(1, -1)$.

slope $= f'(1) = 4$

$y + 1 = 4(x - 1)$

$y = 4x - 5$

(c) At what point of the graph is the tangent horizontal?  
the tangent line is horizontal when the derivative is 0.

$$\frac{2}{x^2} + 2x = 0$$

$$\frac{1}{x^2} + x = 0$$

$$1 + x^3 = 0$$

$$x^3 = -1$$

$$x = -1$$

$y = f(-1) = 3$

Therefore, the tangent line is horizontal at $(-1, 3)$.

4. (12 points) Compute the derivatives of

(a) $f(x) = e^x \cos x + x^2 \sqrt{x} - 7 \tan x + \frac{1}{x} = e^x \cos x + x^{\frac{7}{2}} - 7 \tan x + x^{-1}$

$f'(x) = e^x \cos x + 3x (- \sin x) + \frac{7}{2} x^{\frac{4}{2}} - 7 \sec^2 x - x^{-2}$

(b) $g(x) = \sqrt{x} - 2x + 4$

$g'(x) = \frac{(\frac{1}{2}x^{-\frac{1}{2}} - 2)(x^3 + 12) - (\sqrt{x} - 2x + 4)(3x^2)}{(x^3 + 12)^2}$

(c) $h(x) = \frac{2 \sin x}{\tan x - 4 \cos x}$

$h'(x) = \frac{2 \cos x (\tan x - 4 \cos x) - 2 \sin x (\sec^2 x + 4 \sin x)}{(\tan x - 4 \cos x)^2}$
5. **(8 points)** A particle moves along a straight line and its position at time \( t \) is \( s(t) = t^3 - 9t^2 + 15t + 10 \)

   (a) Find the velocity of the particle at time \( t = 2 \).
   
   \[
   v(t) = s'(t) = 3t^2 - 18t + 15
   \]
   
   \[
   v(2) = 3 \cdot 2^2 - 18 \cdot 2 + 15 = -9
   \]

   (b) When is the particle at rest?
   
   The particle is at rest when \( v(t) = 0 \):
   
   \[
   3t^2 - 18t + 15 = 0
   \]
   
   \[
   3(t^2 - 6t + 5) = 0
   \]
   
   \[
   3(t - 1)(t - 5) = 0
   \]
   
   \[ t = 1 \quad \text{and} \quad t = 5 \]

6. **(6 points)** The volume of a cube with side \( s \) is \( V(s) = s^3 \). What is the rate of change of the volume with respect to \( s \) when \( s = 5 \)?

   The rate of change is the derivative: \( V'(s) = 3s^2 \)
   
   \[
   V'(5) = 3 \cdot 5^2 = 75
   \]

7. **(12 points)** Compute the limits:

   (a) \[
   \lim_{x \to 0} \frac{\cot 3x}{\csc x} = \lim_{x \to 0} \frac{\cos 3x / \sin 3x}{1 / \sin x} = \lim_{x \to 0} \frac{(\sin x)(\cos 3x)}{3 \sin 3x} = \frac{1 \cdot 1}{3 \cdot 1} = \frac{1}{3}
   \]

   (b) \[
   \lim_{x \to 0} \frac{\sin^2 x}{2x} = \lim_{x \to 0} \left( \frac{\sin x \cdot \sin x}{x} \right) = 0 \cdot 1 = 0
   \]

   (c) \[
   \lim_{x \to 0} \frac{\cos x - 1}{x^2 + 4x} = \lim_{x \to 0} \left( \frac{\cos x - 1}{x} \cdot \frac{1}{x + 4} \right) = 0 \cdot \frac{1}{4} = 0
   \]

8. **(10 points)** Where is the function \( f \) differentiable?

   \[
   f(x) = \begin{cases} 
   x + 4 & , x \leq 2 \\
   x^2 - 2x + 6 & , x > 2 
   \end{cases}
   \]

   Both \( x + 4 \) and \( x^2 - 2x + 6 \) are differentiable on \( \mathbb{R} \) (in fact, all polynomials are differentiable on \( \mathbb{R} \)). Therefore, the function \( f \) is differentiable at all points except possibly at 2. So we only have to find out whether \( f \) is differentiable at 2. There are two ways to do this.
1. By definition, $f$ is differentiable at 2 if $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ exists.

\[ f(2) = 6. \] If $x < 2$, then $f(x) = x + 4$. If $x > 2$, then $f(x) = x^2 - 2x + 6$, so we have to find the one-sided limits separately.

\[
\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{x + 4 - 6}{x - 2} = \lim_{x \to 2^-} \frac{x - 2}{x - 2} = 1
\]

\[
\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x^2 - 2x + 6 - 6}{x - 2} = \lim_{x \to 2^+} \frac{x(x - 2)}{x - 2} = \lim_{x \to 2^+} x = 2
\]

The one-sided limits are not equal, therefore $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ doesn’t exist.

Note: You can use the definition $f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$ if you prefer. Consider cases $h < 0$ and $h > 0$, and you will see that $\lim_{h \to 0^+}$ and $\lim_{h \to 0^-}$ are not equal. Therefore, $\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$ doesn’t exist.

2. $f(x)$ is continuous at 2 if and only if
(a) the functions $x + 4$ and $x^2 - 2x + 6$ agree at 2
(b) the derivatives of $x + 4$ and $x^2 - 2x + 6$ agree at 2

Note: the first condition ensures that $f(x)$ is continuous, and the second condition ensures that $f(x)$ is smooth.

Check: (a) $2 + 4 = 2^2 - 2 \cdot 2 + 6$ true
(b) the derivatives are 1 and $2x - 2$, and $1 \neq 2 \cdot 1 - 2$.

So $f(x)$ is not differentiable at 2.

Therefore, the function $f(x)$ is differentiable at all points except for 2.

9. (12 points) Find the derivatives of:

(a) $f(x) = \sqrt{\tan x + 2x}$
   
   $f'(x) = \frac{1}{2}(\tan x + 2x)^{-\frac{1}{2}} \cdot (\sec^2 x + 2)$

(b) $g(x) = \sin^2(\cos x) = (\sin(\cos x))^2$
   
   $g'(x) = 2 \sin(\cos x) \cdot \cos(\cos x) \cdot (-\sin x)$

(c) $h(x) = 2x^2 + 3 \sin x$
   
   $h'(x) = 2 \cdot 2x + 3 \sin x \cdot (2x + 3 \cos x)$

(d) $k(x) = \cos(e^{x}) = \cos(e^{(x-1)})$
   
   $k'(x) = -\sin(e^{(x-1)}) \cdot e^{(x-1)} \cdot (-x^{-2})$