MATH 141
Midterm 2 - Answer key

April 5, 2001

1. **(20 points)** Compute the limits:

   (a) $$\lim_{x\to-\infty}\frac{6x^2+5x}{(1-x)(2x-3)}=\lim_{x\to-\infty}\frac{6x^2+5x}{-2x^2+5x-3}=$$
   $$=\lim_{x\to-\infty}\frac{6+\frac{5}{x}}{-2+\frac{5}{x}-\frac{3}{x^2}}=\frac{6}{-2}=-3$$

   (b) $$\lim_{x\to\infty}\frac{x^2+4x+1}{x^5-3x^3+4}=\lim_{x\to\infty}\frac{\frac{1}{x^3}+\frac{4}{x^4}+\frac{1}{x^5}}{1-\frac{3}{x^2}+\frac{4}{x^5}}=\frac{0}{1}=0$$

   (c) $$\lim_{x\to\infty}\sqrt{x^2+2x+2}-x=\lim_{x\to\infty}\frac{(\sqrt{x^2+2x+2}-x)(\sqrt{x^2+2x+2}+x)}{\sqrt{x^2+2x+2}+x}=$$
   $$=\lim_{x\to\infty}\frac{2x+2}{\sqrt{x^2+2x+2}+x}=\lim_{x\to\infty}\frac{2+\frac{2}{x}}{\frac{\sqrt{x^2+2x+2}}{x}+1}=$$
   $$=\lim_{x\to\infty}\frac{2+\frac{2}{x}}{\sqrt{\frac{x^2+2x+2}{x^2}}+1}==\lim_{x\to\infty}\frac{2+\frac{2}{x}}{\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}+1}=\frac{2}{\sqrt{1}+1}=1$$

   (d) $$\lim_{x\to\infty}(x+\sqrt{x})(x^2+4)=(\infty+\infty)(\infty+4)=\infty$$

   (e) $$\lim_{x\to-\infty}e^{-\frac{3}{x^2}}=e^{\left(\lim_{x\to-\infty}-\frac{3}{x^2}\right)}=e^0=1$$

2. **(10 points)** Find the horizontal asymptotes of the graph of the function $f(x)=\dfrac{x-3}{\sqrt{x^2+3x+2}}$

   $$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1-\frac{3}{x}}{\left(\frac{\sqrt{x^2+3x+2}}{x}\right)}=\lim_{x\to\infty}\frac{1-\frac{3}{x}}{\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}}=1$$

   $$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1-\frac{3}{x}}{\left(\frac{\sqrt{x^2+3x+2}}{x}\right)}=\lim_{x\to-\infty}\frac{1-\frac{3}{x}}{-\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}}=-1$$

   Therefore, the horizontal asymptotes are $y=1$ and $y=-1$.

3. **(10points)** Let $f(x) = x^2 - \frac{2}{x}$.

(a) Find $f'(x)$ using the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - \frac{2}{x+h} - (x^2 - \frac{2}{x})}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{2}{x} - \frac{2}{x+h} + x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{\frac{2x+2h-2x}{x(x+h)} + 2xh + h^2}{h} =$$

$$= \lim_{h\to 0} \left(\frac{2}{x(x+h)} + 2x + h \right) = \frac{2}{x^2} + 2x$$

(b) Find the tangent to the graph of f at $(1, -1)$.

slope=$f'(1) = 4$

$y + 1 = 4(x - 1)$

$y = 4x - 5$

(c) At what point of the graph is the tangent horizontal?

the tangent line is horizontal when the derivative is 0.

$\frac{2}{x^2} + 2x = 0$

$\frac{1}{x^2} + x = 0$

$1 + x^3 = 0$

$x^3 = -1$

$x = -1$

$y = f(-1) = 3$

Therefore, the tangent line is horizontal at $(-1, 3)$.

4. **(12 points)** Compute the derivatives of

(a) $f(x) = e^x \cos x + x^2 \sqrt[3]{x} - 7\tan x + \frac{1}{x} = e^x \cos x + x^{\frac{7}{3}} - 7\tan x + x^{-1}$

$f'(x) = e^x \cos x + 3^x(-\sin x) + \frac{7}{3}x^{\frac{4}{3}} - 7\sec^2 x - x^{-2}$

(b) $g(x) = \dfrac{\sqrt{x} - 2x + 4}{x^3 + 12}$

$$g'(x) = \frac{(\frac{1}{2}x^{-\frac{1}{2}} - 2)(x^3 + 12) - (\sqrt{x} - 2x + 4)(3x^2)}{(x^3 + 12)^2}$$

(c) $h(x) = \dfrac{2\sin x}{\tan x - 4\cos x}$

$$h'(x) = \frac{2\cos x(\tan x - 4\cos x) - 2\sin x(\sec^2 x + 4\sin x)}{(\tan x - 4\cos x)^2}$$

5. **(8 points)** A particle moves along a straight line and its position at time t is $s(t) = t^3 - 9t^2 + 15t + 10$

   (a) Find the velocity of the particle at time $t = 2$.
   $v(t) = s'(t) = 3t^2 - 18t + 15$
   $v(2) = 3 \cdot 2^2 - 18 \cdot 2 + 15 = -9$

   (b) When is the particle at rest?
   The particle is at rest when $v(t) = 0$:
   $3t^2 - 18t + 15 = 0$
   $3(t^2 - 6t + 5) = 0$
   $3(t-1)(t-5) = 0$
   $t = 1 \quad \text{and} \quad t = 5$

6. **(6 points)** The volume of a cube with side s is $V(s) = s^3$. What is the rate of change of the volume with respect to s when $s = 5$?

   The rate of change is the derivative: $V'(s) = 3s^2$

   $V'(5) = 3 \cdot 5^2 = 75$

7. **(12 points)** Compute the limits:

   (a) $$\lim_{x\to 0} \frac{\cot 3x}{\csc x} = \lim_{x\to 0} \frac{\cos 3x / \sin 3x}{1/\sin x} = \lim_{x\to 0} \frac{(\sin x)(\cos 3x)}{\sin 3x} =$$
   $$= \lim_{x\to 0} \frac{\frac{\sin x}{x}(\cos 3x)}{\frac{\sin 3x}{x}} == \lim_{x\to 0} \frac{\frac{\sin x}{x}(\cos 3x)}{3 \cdot \frac{\sin 3x}{3x}} = \frac{1 \cdot 1}{3 \cdot 1} = \frac{1}{3}$$

   (b) $$\lim_{x\to 0} \frac{\sin^2 x}{2x} = \lim_{x\to 0} \left(\frac{\sin x}{2} \cdot \frac{\sin x}{x} \right) = 0 \cdot 1 = 0$$

   (c) $$\lim_{x\to 0} \frac{\cos x - 1}{x^2 + 4x} = \lim_{x\to 0} \left(\frac{\cos x - 1}{x} \cdot \frac{1}{x+4} \right) = 0 \cdot \frac{1}{4} = 0$$

8. **(10 points)** Where is the function f differentiable?

   $$f(x) = \begin{cases} x + 4 & , x \leq 2 \\ x^2 - 2x + 6 & , x > 2 \end{cases}$$

   Both $x + 4$ and $x^2 - 2x + 6$ are differentiable on $\mathbb{R}$ (in fact, all polynomials are differentiable on $\mathbb{R}$). Therefore, the function f is differentiable at all points except possibly at 2. So we only have to find out whether f is differentiable at 2. There are two ways to do this.

1. By definition, f is differentiable at 2 if $f'(2) = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ exists.

$f(2) = 6$. If $x < 2$, then $f(x) = x+4$. If $x > 2$, then $f(x) = x^2 - 2x + 6$, so we have to find the one-sided limits separately.

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{x + 4 - 6}{x - 2} = \lim_{x \to 2^-} \frac{x - 2}{x - 2} = 1$$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x^2 - 2x + 6 - 6}{x - 2} = \lim_{x \to 2^+} \frac{x(x - 2)}{x - 2} = \lim_{x \to 2^+} x = 2$$

The one-sided limits are not equal, therefore $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ doesn't exist.

Note: You can use the definition $f'(2) = \lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$ if you prefer. Consider cases $h < 0$ and $h > 0$, and you will see that $\lim_{h \to 0^-}$ and $\lim_{h \to 0^+}$ are not equal. Therefore, $\lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$ doesn't exist.

2. $f(x)$ is continuous at 2 if and only if
(a) the functions $x + 4$ and $x^2 - 2x + 6$ agree at 2
(b) the derivatives of $x + 4$ and $x^2 - 2x + 6$ agree at 2

Note: the first condition ensures that $f(x)$ is continuous, and the second condition ensures that $f(x)$ is smooth.

Check: (a) $2 + 4 = 2^2 - 2 \cdot 2 + 6$ true
(b) the derivatives are 1 and $2x - 2$, and $1 \neq 2 \cdot 1 - 2$.

So $f(x)$ is not differentiable at 2.

Therefore, the function $f(x)$ is differentiable at all points except for 2.

9. **(12 points)** Find the derivatives of:

(a) $f(x) = \sqrt{\tan x + 2x}$ $\qquad f'(x) = \frac{1}{2}(\tan x + 2x)^{-\frac{1}{2}} \cdot (\sec^2 x + 2)$

(b) $g(x) = \sin^2(\cos x) = (\sin(\cos x))^2$
$g'(x) = 2\sin(\cos x) \cdot \cos(\cos x) \cdot (-\sin x)$

(c) $h(x) = 2^{x^2 + 3\sin x}$ $\qquad h'(x) = \ln 2 \cdot 2^{x^2 + 3\sin x} \cdot (2x + 3\cos x)$

(d) $k(x) = \cos\left(e^{\frac{1}{x}}\right) = \cos\left(e^{(x^{-1})}\right)$
$k'(x) = -\sin\left(e^{(x^{-1})}\right) \cdot e^{(x^{-1})} \cdot (-x^{-2})$