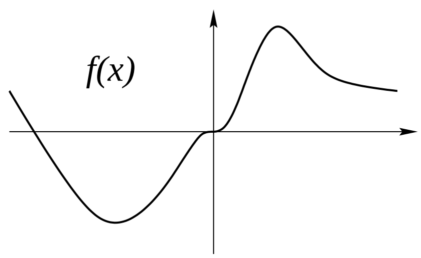
MATH 141

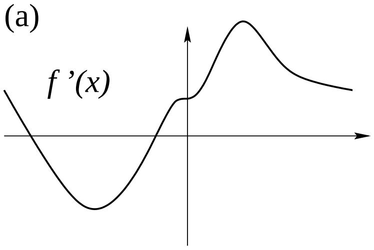
ANSWER KEY TO FINAL EXAM

May 1st, 2000

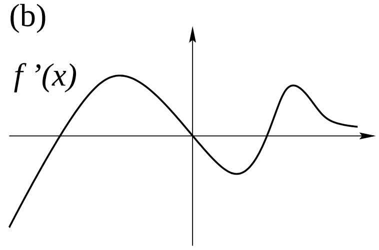
1. **(10pts)** To the right is the graph of $f(x)$. Which of the following best represents the graph of $f'(x)$, the derivative of f?



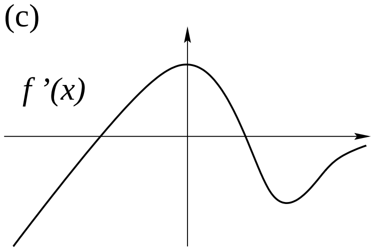
(a)



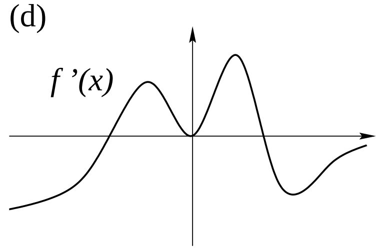
(b)



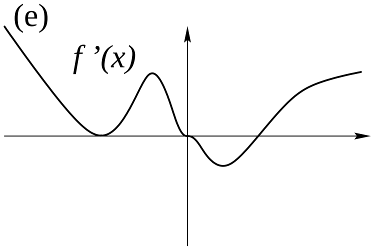
(c)



(d)



(e)



(f) None of the above.

ANSWER: (d)

2. **(8pts)** The slope of the line perpendicular to the line $2x + 3y = 1$ is:

(a) $-\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{3}{2}$ (d) 2

(e) $\frac{1}{2}$ (f) $-\frac{1}{2}$ (g) 3 (h) $\frac{2}{3}$

ANSWER: (c)

3. **(11pts)**

(a) Write down the definition of the derivative of $f(x) = \sin x$ at $x = a$ (as the limit of a difference quotient).

ANSWER: $f'(a) = \lim_{x \to a} \frac{\sin(x) - \sin(a)}{x - a}$

(b) Find the limit from (a) to determine $f'(0)$. Show your working.

ANSWER: $f'(0) = \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

4. **(10pts)** If $y = \cos(x^2 + 1)$, find:

(a) $\frac{dy}{dx}$

ANSWER: $-2x \sin(x^2 + 1)$

(b) $\frac{d^2y}{dx^2}$

ANSWER: $-4x^2 \cos(x^2 + 1) - 2\sin(x^2 + 1)$

5. **(10pts)** If $x \cos y + y \cos x = 1$, find an expression for $\frac{dy}{dx}$.

$$\cos(y) - x\sin(y)y' + y'\cos(x) - y\sin(x) = 0$$

ANSWER: $y' = \frac{y\sin(x) - \cos(y)}{\cos(x) - x\sin(y)}$

6. **(15pts)** For each of the following, find $\dfrac{dy}{dx}$:

(a) $y = \sin(e^{x^2})$

ANSWER: $\cos(e^{x^2})e^{x^2}(2x)$

(b) $y = \dfrac{x^2}{\sin x}$

ANSWER: $\dfrac{2x\sin(x) - x^2\cos(x)}{(\sin(x))^2}$

(c) $y = x\ ln(2)$

ANSWER: $ln(2)$

7. **(18pts)** Determine the following limits (Note: the limit may be a number, ∞, $-\infty$, or may not exist) :

(a) $\displaystyle\lim_{x\to 1} \frac{x - \frac{1}{x}}{x - 1}$

$$\frac{x - \frac{1}{x}}{x - 1}\frac{x}{x} = \frac{x^2 - 1}{(x-1)x} = \frac{x+1}{x}$$

ANSWER: 2

(b) $\displaystyle\lim_{x\to\infty} \frac{3x^2 + 2x - 5}{1 - 3x - x^2}$

ANSWER: -3

(c) $\displaystyle\lim_{x\to 2^-} \frac{|x-2| + 1}{x - 2}$

$$\lim_{x\to 2^-} \frac{|x-2| + 1}{x - 2} = \lim_{x\to 2^-} \frac{3 - x}{x - 2} = -\infty$$

ANSWER: $-\infty$

8. **(10pts)** At what values of x is the function f below discontinuous?

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 3-x & \text{if } x \geq 1 \end{cases}$$

(a) -1 (b) 0 (c) 1 (d) $-1,\ 0$

(e) $0,\ 1$ (f) $-1,\ 1$ (g) $-1,\ 0,\ 1$ (h) Continuous everywhere.

ANSWER: (c)

9. **(8pts)** A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Find the value of t at which the acceleration is equal to zero.

(a) $-\frac{2}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

(e) $-\frac{1}{2}$ (f) $\frac{1}{2}$ (g) $-\frac{3}{2}$ (h) $\frac{3}{2}$

ANSWER: (b)

10. **(5pts)** Find the derivative of $y = (\sin x)^{2x}$

ANSWER: $(2x\cot x + 2ln(\sin x))(\sin x)^{2x}$

11. **(10pts)** Determine the following limits (Note: the limit may be a number, ∞, $-\infty$, or may not exist) :

(a) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x^2 + x}$

$$\lim_{x\to 0} \frac{e^x - 1}{x^2 + x} = \lim_{x\to 0} \frac{e^x}{2x+1}$$

ANSWER: 1

(b) $\lim\limits_{x\to 0+} \dfrac{e^{1/x}}{\ln x}$

$$\lim_{x\to 0+} \frac{e^{1/x}}{\ln x} = \lim_{x\to 0+} \frac{e^{1/x}(-\frac{1}{x^2})}{\frac{1}{x}} = -\lim_{x\to 0+} \frac{e^{1/x}}{x}$$

ANSWER: $-\infty$

12. **(8pts)** If $x^2 + xy + y^2 = 7$, find the value of $\frac{dy}{dx}$ at the point $(1, 2)$.

(a) $-\frac{3}{5}$ (b) $-\frac{3}{4}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

(e) $-\frac{4}{5}$ (f) $\frac{3}{4}$ (g) 1 (h) 0

$$2x + y + xy' + 2yy' = 0 \implies y = -\frac{2x + y}{x + 2y}$$

ANSWER: (e)

13. **(8pts)** Find the critical points of $f(x) = x\sqrt{1 - x^2}$ on the interval $[-1, 1]$.

ANSWER: $-\frac{1}{\sqrt{2}}$, $+\frac{1}{\sqrt{2}}$

14. **(20pts)** Let $f(x) = 3 + 8x^3 + x^4$.

(a) Find the critical points for f.

ANSWER: 0 , -6

(b) Determine the interval(s) on which f is increasing and decreasing.

ANSWER: f is decreasing on $(-\infty, -6)$ and increasing on $(-6, +\infty)$

(c) Determine the interval(s) on which f is concave up and concave down.

ANSWER: f is concave up on $(-\infty, -4)$ and $(0, +\infty)$ and concave down on $(-4, 0)$

(d) Determine the inflection point(s) of f, or state that there are none if none exist.

ANSWER: f has an inflection point at $x = 0$

(e) Where does f have local maxima or minima? State whether a maximum or minimum for each one that you find.

ANSWER: f has a global minimum at -6

15. **(13pts)** Bob is standing near the top of a ladder 15 feet long which is leaning against a vertical wall of his house. The little boy next door ties a rope to the bottom of the ladder and starts to pull the foot of the ladder away from the house wall. The bottom end of the ladder begins to slide away from the wall at the rate of 1 foot per second.

How fast is the top of the ladder sliding down the wall when the foot of the ladder is 9 feet from the wall?

$$x^2 + y^2 = 15^2 \implies 2xx' + 2yy' = 0 \implies yy' = -xx' \implies y' = -\frac{xx'}{y}$$

Note that $x = 9$ and $x' = 1$. Also, $y = 12$ since $9^2 + 12^2 = 15^2$.

$$y' = -\frac{3}{4}$$

ANSWER: $+\frac{3}{4}$

16. **(8pts)** Let $f(x) = x^{2/3}$.

(a) Find a linear approximation of f at $x = 8$.

ANSWER: $4 + \frac{x-8}{3}$

(b) Use this linear approximation to estimate the value of the function at $x = 7$.

ANSWER: $\frac{11}{3}$

17. **(8pts)** The *mean value theorem* states: if $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$ then there is (at least one) c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Find the "c" which the mean value theorem guarantees will exist for the function $f(x) = x^3$ on the interval $1 \leq x \leq 3$.

$$f(3) - f(1) = f'(c)(3-1) \implies f'(c) = 13$$

$$f'(x) = 3x^2 \implies 3c^2 = 13 \implies c = +\sqrt{\frac{13}{3}}, -\sqrt{\frac{13}{3}}$$

NOTE: The negative value is not in the interval $[1, 3]$.

ANSWER: $c = +\sqrt{\frac{13}{3}}$

18. **(10pts)** The length of a rectangle is decreasing at the rate of 1 foot per second, but the area remains constant. How fast is the rectangle's width increasing when its length is 10 feet and its width is 5 feet.

(a) $\frac{1}{2}$ (b) 4 (c) $\frac{1}{10}$ (d) 2

(e) 5 (f) $\frac{1}{4}$ (g) 10 (h) $\frac{1}{5}$

$$A = lw \implies 0 = l'w + lw' \implies w' = -\frac{l'}{l}w$$

Note that $l = 10$, $l' = -1$, and $w = 5$.

ANSWER: (a)

19. **(10pts)** The radius of a circle is given as 10 cm, with a possible error of measurement equal to 1 mm. Use differentials to estimate the maximum error in the **area** (in cm^2).

(a) 10π (b) 2π (c) 3π (d) π

(e) 8π (f) 5π (g) 6π (h) 4π

$$A = \pi r^2 \implies dA = 2\pi r dr$$

Note that $r = 10$ and $dr = 0.1$.

ANSWER: (b)