Math Field Day Prep Session
Grades 9-10

Number Theory

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Useful formulas and theorems, and hints

Know basic terminology: natural numbers, integers, rational numbers, real numbers; factor, divisor, multiple, divides, is divisible by.

$1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$, proof by Gauss, sum of any arithmetic sequence.

Partial sum of geometric series: $1 + q + q^2 + \cdots + q^n = \frac{1-q^{n+1}}{1-q}$.

Sum of infinite geometric series: $1 + q + q^2 + \cdots = \frac{1}{1-q}$.

Difference of squares/cubes, sum of cubes, square/cube of sum/difference, etc.

Prime Factorization theorem: every positive integer larger than 1 can be written as a product of primes, uniquely up to order. Know the prime factorization of the current year off the top of you head.

The number of positive divisors of $N = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$ is $(k_1 + 1)(k_2 + 1) \dots (k_n + 1)$.

Divisibility tests for 2, 3, 4, 5, 8, 9, 10, 11.

May need to introduce notations, e.g. for n, $n+1$, $n+2$, etc. for consecutive integers, $2n/2n+1$ for even/odd integers, etc.

May need to perform some calculations, up to multiplying/dividing 2- or 3-digit numbers, or adding/subtracting 4-digit numbers, but usually not larger ones. Look for ways to simplify your calculations: factor, reduce fractions, etc.

Examples

1. Compute: $1 + 2 + 3 + \cdots + 2019$.

   Solution. $1 + 2 + 3 + \cdots + 2019 = \dfrac{2019 \cdot 2020}{2} = 2019 \cdot 1010 = 2039190.$

2. If $a = 6$ and $b = 24$, find $\dfrac{a^8 - b^4}{(a^4 + b^2)(a^2 + b)}$.

   Solution.
   $$\frac{a^8 - b^4}{(a^4 + b^2)(a^2 + b)} = \frac{(a^4 - b^2)(a^4 + b^2)}{(a^4 + b^2)(a^2 + b)} = \frac{(a^2 - b)(a^2 + b)(a^4 + b^2)}{(a^4 + b^2)(a^2 + b)} = a^2 - b = 36 - 24 = 12.$$

3. How many different positive factors does the number 10! have?

   Solution. $10! = 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$
   The number of positive factors is $9 \cdot 5 \cdot 3 \cdot 2 = 270$.

4. Today is Thursday. What day of the week will be exactly 2019 days from today?

   Solution. $2019 = 288 \cdot 7 + 3$. Answer: Sunday.

5. For how many positive integers m does there exist at least one integer n such that $mn \leq m+n$?

   Solution. For any positive integer m, $n = 0$ works since $0 \leq m$. Answer: infinitely many.

Problems

Mad Hatter

1. (MH 9-10, 2006) Determine the sum $3 + 7 + 11 + \cdots + 35$.

   (a) 140

   (b) 171

   (c) 315

   (d) 342

   Solution. Let

   $$\begin{aligned} S &= 3 + 7 + 11 + \cdots + 35 \\ S &= 35 + 31 + 27 + \cdots + 3 \\ 2S &= 38 + 38 + 38 + \cdots + 38 = 38 \cdot 9 = 342. \\ S &= 171 \end{aligned}$$

   Answer: (b)

2. (MH 9-10, 2014) Find the first of three consecutive odd integers whose sum is 57.

   (a) 13

   (b) 15

   (c) 17

   (d) 19

   Solution. If $2n + 1$, $2n + 3$, $2n + 5$ are three consecutve odd integers, their sum is $6n + 9$.

   Solving $6n + 9 = 57$ gives $n = 8$. So the first number is $2n + 1 = 17$.

   Alternatively, we can write the three consecutive odd integers as $a - 2$, a, and $a + 2$, then their sum is $(a - 2) + a + (a + 2) = 3a$, i.e. 3 times the middle number. Thus the middle number is $57/3 = 19$. The first number is then 17.

   Finally, we can just check the given answer choices:

   $13 + 15 + 17 \neq 57$,

   $15 + 17 + 19 \neq 57$,

   $17 + 19 + 21 = 57$.

   Answer: (c)

3. (MH 9-10, 2006) Suppose a positive integer N is divisible by both 9 and 21. What is the smallest possible number of positive integers that divide N?

   (a) 6

   (b) 5

   (c) 4

   (d) 3

Solution. N is divisible by both 3^2 and $3 \cdot 7$, so its prime factorization must contain at least $3^2 \cdot 7$. It will have the smallest possible number of positive divisors if $N = 3^2 \cdot 7$. This number has $3 \cdot 2 = 6$ positive divisors.

Answer: (a)

4. (MH 11-12, 2010) Which of the following numbers is a perfect square?

   (a) 98!99!

   (b) 98!100!

   (c) 99!100!

   (d) 99!101!

   (e) 100!101!

   Solution.

   $98!99! = (98!)^2 \cdot 99$ – not a perfect square

   $98!100! = (98!)^2 \cdot 99 \cdot 100$ – not a perfect square

   $99!100! = (99!)^2 \cdot 100 = (99!)^2 \cdot 10^2 = (99! \cdot 10)^2$

   Also, answers (d) and (e) are not perfect squares (can verify this similarly to (a) and (b)).

   Answer: (c)

5. (MH 9-10, 2006) What is the smallest positive prime p greater than 2 such that $p^3 + 7p^2$ is a perfect square?

   (a) 13

   (b) 17

   (c) 23

   (d) 29

   Solution. We have $p^3 + 7p^2 = p^2(p + 7)$, so $p + 7$ must be a perfect square. Checking values $p+7 = 9, 16, 25, 36$ we see that the smallest such prime p greater than 2 is 29. Alternatively, check answers: For $p = 13$, $p+7 = 20$ is not a perfect square. For $p = 17$, $p+7 = 24$, is not a perfect square. Also, for $p = 23$, $p + 7 = 30$ is not a perfect square again. Finally, for $p = 29$, we get $p + 7 = 36$ which is a perfect square.

   Answer: (d)

6. (MH 9-10, 2017) Let n be a positive integer. If n is divided by 2, 3, 4, 5, or 6, the remainder is 1, but n is divisible by 7. What is the least possible value of n?

   (a) 421

   (b) 721

   (c) 301

   (d) 63

   (e) None of the above

   Solution. Since $n - 1$ is divisible by 2, 3, 4, 5, and 6, it is a multiple of the LCM of these numbers, which is 60. So n is 1 larger than a multiple of 60. Checking such numbers, that is, 61, 121, 181, 241, 301, $\ldots$, we see that 301 is the smallest that is divisible by 7.

   Answer: (c)

7. (MH 9-10, MH 11-12, 2006) Evaluate: $\dfrac{4351^2-4347^2}{4350\cdot 4353-4351^2}$.

   (a) $\dfrac{1}{2}$
   (b) 1
   (c) 2
   (d) 4
   (e) 8

   Solution. Let $4350 = n$. Then
   $$\frac{4351^2-4347^2}{4350\cdot 4353-4351^2} = \frac{(n+1)^2-(n-3)^2}{n(n+3)-(n+1)^2} = \frac{n^2+2n+1-n^2+6n-9}{n^2+3n-n^2-2n-1} = \frac{8n-8}{n-1} = 8.$$
   Answer: (e)

8. (MH 9-10, 2017) Which of the following CANNOT be the sum of the digits of a square?

   (a) 13
   (b) 11
   (c) 7
   (d) 4

   Solution. One approach is to check the sums of the digits for some small squares: 1, 4, 9, 16, 25, 36, 49, ... We see that 4, 7, and 13 already appear. Thus the answer must be 11.

   Another approach is to observe that if a number is divisible by 3, then its square is divisible by 3, and therefore the sum of the digits of the square is also divisible by 3. However, if a number is not divisible by 3, then it can be written in the form $3n+1$ or $3n+2$, so its square is $(3n+1)^2 = 9n^2+6n+1$ or $(3n+2)^2 = 9n^2+12n+4$. We see that in either case the square has a remainder of 1 upon division by 3. It follows that the sum of the digits also has remainder of 1 upon division by 3. Since 11 has remainder 2, it cannot be the sum of the digits of a square.

   Answer: (b)

9. (MH 9-10, 2017) What is the tens digit of the smallest positive integer that is divisible by each of 20, 16 and 2016?

   (a) 0
   (b) 2
   (c) 4
   (d) 8

   Solution. First we find the prime factorization of 2016: $2^5\cdot 3^2\cdot 7$. So the number we are looking for is the LCM of $2^2\cdot 5$, 2^4, and $2^5\cdot 3^2\cdot 7$. The answer is $2^5\cdot 3^2\cdot 5\cdot 7 = 2016\cdot 5 = 10080$.

   Answer: (d)

10. (MH 9-10, 2017) What is the smallest positive integer $x > 100$ such that every permutation of the digits of x is prime?

    (a) 101
    (b) 103
    (c) 113
    (d) 117

Solution. Since 110 and 130 are not prime, we can easily eliminate answers (a) and (b). Next, 117 is divisible by 3, so is not prime. Only 113 remains, so this must be the answer. Note: it can be checked that 113, 131, and 311 are all prime, however, we do not have to do this since we have eliminated all other answer choices.

Answer: (c)

Leap Frog

1. (LF 9-12, 2006) The units digit of the number $9^{2006} - 3^{2006}$ is

   (a) 6
   (b) 4
   (c) 2
   (d) 0
   (e) None of these

   Solution. Since $9^2 = 81$ and the units digit of any power of 81 is 1,
   $9^{2006} = (9^2)^{1003}$ has units digit 1, and
   $3^{2006} = (3^2)^{1003} = 9^{1003} = 9 \cdot 9^{1002} = 9 \cdot (9^2)^{501}$ has units digit 9.
   The units digit of the difference is 2.

   Answer: (c)

2. (LF 9-10, 2013) How many 4-digit palindromic numbers *abba* are divisible by 9?

   (a) 7
   (b) 8
   (c) 9
   (d) 10
   (e) None of these

   Solution.

   $9|abba$ iff $9|2(a+b)$ iff $9|(a+b)$.

   Since $a \neq 0$, $a+b=9$ or $a+b=18$.

   $a+b=9$ has 9 solutions: $(1,8)$, $(2,7)$, …, $(9,0)$.

   $a+b=18$ has 1 solution: $(9,9)$.

   Answer: (d)

3. (LF 9-10, 2015) Suppose that when dividing the number n by 7, there results a remainder of 3. What then is the remainder if you were to divide the number $2015n$ by 7?

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) None of these

Solution. Since dividing n by 7 results in a remainder of 3, we have $n = 7q + 3$ for some integer $q \geq 0$. Thus, $2015n = 2015 \cdot 7q + 2015 \cdot 3 = 7(2015q) + 6045$. Dividing 6045 by 7 results in a quotient of 863 with a remainder of 4, $6045 = 7 \cdot 863 + 4$. Thus, $2015n = 7(2015q) + 7 \cdot 863 + 4 = 7(2015q + 863) + 4 = 7q_0 + 4$, where $q_0 = 2015q + 863$. This means that dividing $2015n$ by 7 results in a remainder of 4, none of the answer choices presented.

Answer: (e)

4. (MH 9-10, 2006) If a, b, $a + b$, and $a - b$ are all prime numbers, which of the following statements must be true about the sum of these four numbers?

   (a) The sum is odd and prime.

   (b) The sum is odd and divisible by 3.

   (c) The sum is odd and divisible by 7.

   (d) The sum is even.

   Solution. The sum is $a + b + a + b + a - b = 3a + b$. None of the above is obviously true or false from this expression, so let's try to find some information about a and b.
   If both a and b are odd, then $a + b$ is even and at least 6, so not prime.
   If $a = b = 2$, then $a + b = 4$ and is not prime.
   Then a is an odd prime, and $b = 2$. So $3a + b = 3a + 2$ where a is odd. This eliminates answer choices (b) and (d).
   Further, $a \neq 3$ since $3 - 2 = 1$ is not prime. But $a = 5$ gives $a - b = 3$ and $a + b = 7$, so all four numbers are prime. Thus $3a + b = 17$, and this eliminates (c).
   Alternatively, one of a, $a + 2$, $a - 2$ is divisible by 3. Since it is also prime, it must be equal to 3. So $a - 2 = 3$, and $a = 5$. Then proceed as above.

   Answer: (a)

5. (LF 9-12, 2006) Suppose n, a and b are positive integers. In order for n to divide ab, it is ________________ that n divides a or n divides b.

   (a) necessary and sufficient

   (b) necessary, but not sufficient

   (c) sufficient, but not necessary

   (d) neither necessary nor sufficient

   (e) None of these

   Solution. It is not necessary because 4 divides $4 = 2 \times 2$, but 4 does not divide 2. It is sufficient, however. This is because if n divides say a, then $a = nm$ for some positive integer m, and hence $ab = nmb$ which is clearly divisible by n.

   Answer: (c)

6. (LF 9-10, 2017) The sum of eight consecutive integers is 212. What is the sum of the first and last integers?

   (a) 52

   (b) 53

   (c) 54

   (d) 55

   (e) None of these

Solution. The sum of $x,\ x+1,\ x+2,\ldots,x+7$ is $8x+28$. To find x, we solve $8x+28=212$ and get $x=23$. The first and last terms are 23 and 30, so their sum is 53. Alternatively, the sum of the first and last of the 8 terms is $\frac{1}{4}$ of the total sum.

Answer: (b)

7. (LF 9-10, 2015) For how many of the ten digits $x=0,1,2,\ldots,9$ is the 2017-digit number $n=1\underbrace{xx...x}_{2015}0$ divisible by 24?

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) None of these

   Solution. Since $24=8\cdot 3$, we must have that n is divisible by 3 and 8. A number is divisible by 3 when its digital sum is divisible by 3. So we must have that $1+2015x$ is divisible by 3. Now, $2015=2013+2$, and so $1+2015x=(1+2x)+2013x=(1+2x)+3(671x)$. Thus, $1+2015x$ is divisible by 3 when $1+2x$ is divisible by 3. The possible vales for x to ensure that n is divisible by 3 are then reduced to $x=1,4,7$. Next, n is divisible by 8 precisely when the 3-digit number $xx0$ is divisible by 8. It is an easy check that of the digits $x=1,4,7$, only $x=4$ satisfies this condition. So there is only one possible solution, $x=4$.

   Answer: (b)

8. (LF 2014 9-10) The digit sum of a number is the sum of its decimal digits. For example, the digit sum of the number 3206 is $3+2+0+6=11$. Determine the digit sum of the number $(10^{2014}+1)^4$.

   (a) 10
   (b) 12
   (c) 14
   (d) 16
   (e) None of these

   Solution. $(10^{2014}+1)^4=(10^{2014})^4+\binom{4}{1}(10^{2014})^3+\binom{4}{2}(10^{2014})^2+\binom{4}{3}10^{2014}+1=$

   $10^{8056}+4\cdot 10^{6042}+6\cdot 10^{4028}+4\cdot 10^{2014}+1.$

   The non-zero digits of this number are 1, 4, 6, 4, 1. Their sum is 16.

   Alternatively, $(10^{2014}+1)^4\equiv(1^{2014}+1)^4\equiv 2^4\equiv 16 \pmod 9$. So the answers (a), (b), (c) can be eliminated. Now we have only (d) and (e) left.

   Answer: (d)

More Problems

1. (MH 9-10, 2006) How many 2-digit numbers, $n\geq 10$ are there such that both digits are squares (e.g., 10 and 41 are two such numbers)?

(a) 3
(b) 6
(c) 8
(d) 12

Solution. The first digit can be 1, 4, or 9. The second digit can be 0, 1, 4, or 9. There are $3 \cdot 4 = 12$ such numbers.

Answer: (d)

2. (MH 11-12, 2006) Simplify: $\dfrac{123 \cdot 456 + 123 + 456}{123 + 456 \cdot 124} =$

   (a) 0.5
   (b) 1
   (c) 2
   (d) $\dfrac{123}{124}$
   (e) None of the above

   Solution. $\dfrac{123 \cdot 456 + 123 + 456}{123 + 456 \cdot 124} = \dfrac{(123+1) \cdot 456 + 123}{123 + 456 \cdot 124} = \dfrac{124 \cdot 456 + 123}{123 + 456 \cdot 124} = 1$

   Answer: (b)

3. (MH 9-10, 2006) Which of the following is the largest number?

   (a) $2^{(3^4)}$
   (b) $4^{(3^2)}$
   (c) $8^{(4^2)}$
   (d) $(16^8)^2$

   Solution. Since all bases are powers of 2, express each number as a power of 2:
   $2^{(3^4)} = 2^{81}$,
   $4^{(3^2)} = 4^9 = (2^2)^9 = 2^{18}$,
   $8^{(4^2)} = 8^{16} = (2^3)^{16} = 2^{48}$,
   $(16^8)^2 = 16^{16} = (2^4)^{16} = 2^{64}$.

   Answer: (a)

4. (MH 11-12, 2006) Find the smallest positive prime p such that $p^3 + 10p^2$ is a perfect square. What is the sum of the digits of p?

   (a) 7
   (b) 8
   (c) 12
   (d) 14
   (e) None of the above

   Solution. $p^3 + 10p^2 = p^2(p + 10)$, so $p + 10$ must be a perfect square. Checking the values $p + 10 = 16, 25, 36, 49, 64, 81$ we see that the smallest prime p that has this property is 71. The sum of its digits is 8.

   Answer: (b)

5. (MH 11-12, 2006) Approximately what percentage of the first $10,000$ natural numbers have a 1 somewhere in them?

   (a) 10%
   (b) 22%
   (c) 34%
   (d) 45%
   (e) None of the above

   Solution. Count the number of integers from 1 to $9,999$ that do not have a 1, i.e. consist entirely of digits 0, 2, 3, ...,, 9, i.e. each of the four digits has 9 choices. There are 9^4 such integers, and $9^4 = (9^2)^2 = 81^2 = 6561$, or approximately 66%. Therefore approximately 34% have a 1 somewhere in them.

   Answer: (c)

6. (MH 11-12, 2006) The sum of seven consecutive numbers is 126. Find the product of the smallest and the largest of these numbers.

   (a) 315
   (b) 324
   (c) 450
   (d) 468
   (e) None of the above

   Solution. $(n-3)+(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3) = 126$
   $7n = 126$
   $n = 18$
   $n-3 = 15$, $n+3 = 21$, $15 \cdot 21 = 315$

   Answer: (a)

7. (MH 11-12, 2006) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{2005\cdot 2006} =$

   (a) $\dfrac{2005}{2}$
   (b) 1003
   (c) $\dfrac{2005}{2006}$
   (d) $1 - \dfrac{1}{2005\cdot 2006}$
   (e) None of the above

   Solution. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{2005\cdot 2006} =$
   $\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots+\left(\dfrac{1}{2005}-\dfrac{1}{2006}\right) = 1-\dfrac{1}{2006} = \dfrac{2005}{2006}$
   Answer: (c)

8. (MH 9-10, 2006) What is $\sqrt{30}$ to the nearest hundredth?

   (a) 5.25
   (b) 5.38

(c) 5.48
(d) 5.77

Solution. $5.4^2 = 29.16$, $5.5^2 = 30.25$, so the answer must be 5.48.

Answer: (c)

9. (MH 9-10, 2006) What is the greatest common divisor of 1776 and 1976?

(a) 16
(b) 8
(c) 4
(d) 2

Solution. Since all answer choices are powers of 2, we only have to check factors that are powers of 2.
$1776 = 2 \cdot 888 = 2 \cdot 8 \cdot 111 = 16 \cdot 111$.
$1976 = 2 \cdot 988 = 4 \cdot 494 = 8 \cdot 247$. So the answer is 8.

Answer: (b)

10. (MH 9-10, 2006) What is the least common multiple of 36 and 243?

(a) 8748
(b) 2916
(c) 972
(d) 108

Solution. $36 = 2^2 \cdot 3^2$, $243 = 3^5$.
$\text{LCM} = 2^2 \cdot 3^5 = 4 \cdot 243 = 972$.

Answer: (c)

11. (MH 9-10, 2006) If three distinct counting numbers have a sum of 10 and a product of 20, what is the median of the three numbers?

(a) 3
(b) 4
(c) 5
(d) There is not enough information given.

Solution. $20 = 2 \cdot 2 \cdot 5$, and all numbers are less than 10 and distinct, the only possibility is 1, 4, and 5. Their sum is indeed 10 (just checking!), and the median is 4.

Answer: (b)

12. (LF 9-12, 2006) Suppose that the average of 10 numbers is 90. One of the numbers in the list is deleted, and the resulting average of nine numbers is equal to 91. What is the value of the deleted number?

(a) 80
(b) 81
(c) 82
(d) 83
(e) None of these

Solution. Let the 10 numbers be denoted by $N_1, N_2, \ldots, N_{10}$, where N_{10} will be the deleted number. We are given that the average of the original 10 numbers is 90. Thus

$$\frac{N_1 + N_2 + \cdots + N_{10}}{10} = 90,$$

which impies $N_1 + N_2 + \cdots + N_{10} = 900$. We are also given that

$$\frac{N_1 + N_2 + \cdots + N_9}{9} = 91,$$

and so $N_1 + N_2 + \cdots + N_9 = 819$. Putting this together gives

$$819 + N_{10} = 900$$

and so $N_{10} = 81$.

Answer: (b)

13. (MH 11-12, 2006) Which of the numbers 2^{100}, 3^{50}, $10,000^2$, 500^3 is the largest?

   (a) 2^{100}
   (b) 3^{50}
   (c) $10,000^2$
   (d) 500^3

   Solution. $2^{100} = (2^2)^{50} = 4^{50} > 3^{50}$. So the answer is not (b).
   $2^{100} = (2^{10})^{10} = (1024)^{10} > (10^3)^{10} = 10^{30}$,
   $10,000^2 = 100,000,000$,
   $500^3 = 125,000,000$.

   Answer: (a).

14. (MH 11-12, 2012) How many different prime factors does the number 20! have?

   (a) 6
   (b) 8
   (c) 19
   (d) 20
   (e) None of the above

   Solution. The prime factors of 20! are all the prime numbers between 1 and 20:

   2, 3, 5, 7, 11, 13, 17, 19. So there are 8 of them.

   Answer: (b)

15. (MH 11-12, 2006) How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 5

Solution. An integer has 15, 20, and 25 as factors when 3, 5^2, and 2^2 its factors, i.e. the integer is divisible by $3 \cdot 5^2 \cdot 2^2 = 3 \cdot 25 \cdot 4 = 300$. Integers between 1000 and 2000 that are divisible by 300 are: 1200, 1500, and 1800. So the answer is 3.

Answer: (c)

16. (MH 11-12, 2006) Simplify: $\dfrac{\sqrt{4}+\sqrt{6}+\sqrt{24}}{\sqrt{1}+\sqrt{6}+\sqrt{9}+\sqrt{150}}$.

   (a) $\dfrac{1}{5}$

   (b) $\dfrac{1}{2}$

   (c) 1

   (d) $\dfrac{4}{15}$

   (e) None of the above

   Solution. $\dfrac{\sqrt{4}+\sqrt{6}+\sqrt{24}}{\sqrt{1}+\sqrt{6}+\sqrt{9}+\sqrt{150}} = \dfrac{2+\sqrt{6}+2\sqrt{6}}{1+\sqrt{6}+3+5\sqrt{6}} = \dfrac{2+3\sqrt{6}}{4+6\sqrt{6}} = \dfrac{1}{2}$.

   Answer: (b)

17. (LF 9-12, 2006) Let $(0.xyxyxy\ldots)_b$ and $(0.yxyxyx\ldots)_b$ be the base b representations of the two numbers A and B respecitvely, where x and y represent base b digits, not both of which are zero. Then,

$$\frac{A}{B} =$$

   (a) $\dfrac{y+b}{x+b}$

   (b) $\dfrac{x+b}{y+b}$

   (c) $\dfrac{xb+y}{yb+x}$

   (d) $\dfrac{yb+x}{xb+y}$

   (e) None of these

   Solution 1. Let's first convert A to base 10.

$$\begin{aligned}
A &= \frac{x}{b}+\frac{y}{b^2}+\frac{x}{b^3}+\frac{y}{b^4}+\cdots \\
&= \left(\frac{x}{b}+\frac{x}{b^3}+\cdots\right)+\left(\frac{y}{b^2}+\frac{y}{b^4}+\cdots\right) \\
&= \frac{x}{b}\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right)+\frac{y}{b^2}\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right) \\
&= \left(\frac{x}{b}+\frac{y}{b^2}\right)\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right) \\
&= \frac{bx+y}{b^2}\cdot\frac{1}{1-\frac{1}{b^2}} \\
&= \frac{bx+y}{b^2-1}
\end{aligned}$$

By reversing the roles of x and y, we get $B = \frac{by+x}{b^2-1}$ and so

$$\frac{A}{B} = \frac{bx+y}{by+x}.$$

Solution 2. From the given base-b representations we get two equations (1) $bA = B + x$ and (2) $bB = A + y$. Equation (1) gives $B = bA - x$, and substituting this into Equation (2) gives

$$\begin{aligned} b(bA - x) = A + y \quad &\Longrightarrow \quad b^2A - bx = A + y \\ &\Longrightarrow \quad (b^2-1)A = bx + y \\ &\Longrightarrow \quad A = \frac{bx+y}{b^2-1}. \end{aligned}$$

Then proceed as in solution 1.

Answer: (c)

18. (LF 9-12, 2006) Suppose N is the 3-digit number $N = abc$ where a, b and c are the respective hundreds, tens and units digits. Assuming that when N^2 is divided by 100 the remainder is 1, then what are the possible values for b?

    (a) $b = 0, 4, 5,$ or 9

    (b) $b = 0, 4, 5, 6,$ or 9

    (c) $b = 0, 4, 5,$ or 8

    (d) $b = 0, 4, 6, 8,$ or 9

    (e) None of these

    Solution.

$$\begin{array}{ccccc} & & a & b & c \\ & \times & a & b & c \\ \hline & & ac & bc & c^2 \\ & ab & b^2 & bc & \\ a^2 & ab & ac & & \\ \hline & \ldots & & 2bc & c^2 \end{array}$$

We first observe that the answer is independent of a. This is because $N = 100a + (bc)$ and so $N^2 = 10000a^2 + 200a(bc) + (bc)^2$. Since $10000a^2 + 200a(bc)$ is divisible by 100, the remainder of N is the same as the remainder of (bc). Thus, in our analysis we may replace N by $N' = bc = 10b + c$. Square N' to get

$$\begin{aligned} N'^2 &= (10b + c)^2 \\ &= 100b^2 + 20bc + c^2. \end{aligned}$$

The $100b^2$ term will not contribute to the remainder, and so may be ignored. We are then left with dividing $20bc + c^2$ by 100 to get a remainder of 1. Clearly this means $c = 1$ or 9.

If $c = 1$, then $20bc + c^2 = 20b + 1$, which by checking the possibilities exhaustively, will have a remainder of 1 only if $b = 0$ or 5. And this can happen if say $N = 101$ or 151.

Similarly, if $c = 9$, then $20bc + c^2 = 180b + 81$, which will have a remainder of 1 only if $b = 4$ or 9. This can happen if say $N = 149$ or $N = 199$.

So the possible values for b are 0, 4, 5 or 9.

Answer: (a)

19. (LF 9-12, 2006) If the natural numbers 0, 1, 2, ... are lined up to form a non-ending string of digits

$$01234567891011121314\ldots$$

then the 2006th digit in the string is

(a) 9
(b) 0
(c) 3
(d) 7
(e) None of these

Solution. The numbers 0 through 9 give 10 digits.
The numbers 10 through 99 give 180 digits.
So the numbers 0 through 99 contribute 190 digits.
We need 1816 more digits.
Since $1816 = 605 \times 3 + 1$, we know we can accomplish this with the 1st digit of the 606th number after 99. This number is 705. So the 2006th digit in the sequence is 7.

Answer: (d)

20. How many positive integer divisors of 1,000,000 are there?

(a) 49
(b) 50
(c) 36
(d) 100
(e) None of these

Solution. Write 1,000,000 as $10^6 = 2^6 \cdot 5^6$. The positive integer divisors are then in the form $2^x \cdot 5^y$ where $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. There are 7 possible solutions for x and 7 possible solutions for y, giving $7 \cdot 7 = 49$ positive divisors.

Answer: (a)