Math Field Day Prep Session
Grades 9-10

Geometry

Department of Mathematics, CSU Fresno

Useful formulas and theorems

Formulas for perimeter, circumference, area, surface area, volume of basic shapes (triangle, rectangle, trapezoid, parallelogram, circle, prism, pyramid, cylinder, cone, sphere), length of an arc of a circle

Similar figures/solids and their perimeters/areas/volumes.

Sum of interior angles in any triangle is $180°$, in any n-gon $(n-2)\cdot 180°$.

Relationships between interior/exterior angles in a triangle.

In a regular n-gon, each exterior angle is $\frac{360°}{n}$, each interior angle is $\frac{n-2}{n}\cdot 180°$.

Ratios of lengths of sides of $30°-60°-90°$ and $45°-45°-90°$ triangles.

The area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$.

Pythagorean Theorem.

The three medians in any triangle are concurrent and each median is divided by the intersection point into two parts whole lengths have ratio 1 : 2.

Examples

1. The radius of a sphere is tripled, by what number is its volume multiplied?

   Equivalent question: One sheet of metal can be melted down to make a spherical ball with a radius of 2 cm. How many such sheets would need to be melted down to make a spherical ball of radius 6 cm?

   Solution. Since the volume grows proportionally to the cube of the radius, the volume increases by a factor of $3^3 = 27$ when the radius increases by a factor of 3.

2. What is the measure of each interior angle of a regular decagon?

   Solution. Each exterior angle is $\frac{1}{10}\cdot 360° = 36°$, so each interior angle is $180° - 36° = 144°$.

Problems

Mad Hatter

1. (MH 9-10 2017) If the measure, in degrees, of the three angles of a triangle are x, $x+10$, and $2x-6$, the triangle must be

   (a) right.

   (b) equilateral.

   (c) isosceles.

   (d) scalene.

Solution. The sum of the interior angles of any $\triangle ABC$ is $180°$. Then, let $m\angle A = x,\ m\angle B = x + 10,$ and $m\angle C = 2x - 6$. Since the sum of the angles is $180°$, then,

$$\begin{aligned} x + x + 10 + 2x - 6 &= 180° \\ 4x + 4 &= 180° \\ x &= 44° \end{aligned}$$

Now, substituting the value of x, we see that $m\angle A = 44°, m\angle B = 54°,$ and $m\angle C = 82°$. Since none of the angles are $90°$, $\triangle ABC$ is not a right triangle. Since the angles are all distinct, $\triangle ABC$ is neither equilateral nor isosceles. Thus, $\triangle ABC$ is scalene.

Answer: (d)

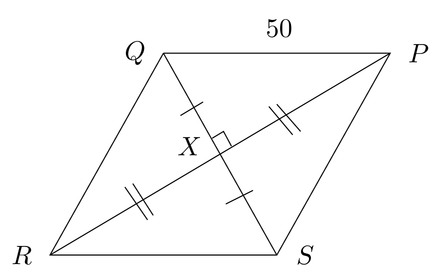
2. (MH 9-10 2017) The perimeter of a rhombus is 200 feet and one of its diagonals is 80 feet. What is the area of the rhombus?

   (a) 1200
   (b) 1500
   (c) 2000
   (d) 2400

   Solution. Let $PQRS$ be a rhombus with perimeter 200 and $PR = 80$. A rhombus has all four sides equal to each other in length, thus the perimeter of 200 divided by 4 gives the side of the rhombus, 50 feet. Let X be the point of intersection of PR and QS. It divides each diagonal in half. Then $PX = 40$. Consider the right triangle PXQ and apply the Pythagorean theorem to find the length of side XQ:
   $PX^2 + XQ^2 = PQ^2$
   $40^2 + XQ^2 = 50^2$
   $1600 + XQ^2 = 2500$
   $XQ^2 = 900$
   $XQ = 30.$

   Now we calculate the area of the right triangle PXQ and multiply it by four to get the area of the rhombus.

   Area $= 4((\text{base} \cdot \text{height})/2) = 4((40 \cdot 30)/2) = 4 \cdot 600 = 2400.$



   Answer: (d)

3. (MH 9-10 2017) An analog clock displays the time 3:40. What is the measure of the smaller angle formed by the minute and hour hands of the clock?

   (a) $100°$
   (b) $110°$
   (c) $120°$

(d) $130°$

Solution. The full angle is $360°$. Since the clock is split into 12 hours, each angle between consecutive hours is $\frac{360°}{12} = 30°$. Let the center of the clock be point O. Since the clock reads 3:40, the minute hand will be exactly at the 8th hour (let us call it point A). The hour hand will be $\frac{40}{60}$ or $\frac{2}{3}$ of the way between hours 3 and 4 (let us call it point B). Then, there are $4 + \frac{1}{3}$ hours between A and B. Since each hour is $30°$, $m\angle AOB = 30° \cdot \frac{13}{3} = 130°$.

Answer: (d)

4. (MH 9-10, 2015) Three balls are stacked in a cylinder that touches the stack on all sides and on the top and bottom. What is the ratio of the volume of balls to the volume of the cylinder?

   (a) $\frac{2}{9}$
   (b) $\frac{2}{3}$
   (c) $\frac{4}{9}$
   (d) $\frac{4}{3}$

   Solution. We need only to look at the ratio for one ball in the cylinder. This is because if each ball has a volume of V_b and each cylinder has a volume of V_c, then 3 balls and 3 cylinders would have a ratio of $\dfrac{3V_b}{3V_c} = \dfrac{V_b}{V_c}$. Then we get $\dfrac{\frac{4}{3}\pi r^3}{2\pi r^3}$, where the numerator is the volume of a sphere and the denominator is the volume of the cylinder. The fraction reduces to $\frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$.

   Answer: (b)

5. (MH 9-10, 2009) The surface area of a large cube is 5400 square inches. This cube is cut into a number of identical smaller cubes, each having a volume of 216 cubic inches. How many smaller cubes are there?

   (a) 180
   (b) 164
   (c) 125
   (d) 64

   Solution. The area of one face of the large cube is $\frac{5400}{6} = 900$ square inches, so each face is 30×30. The size of each small cube is $6 \times 6 \times 6$ (since its volume is $216 = 6^3$), so 5 small cubes fit along each edge of the big one. Therefore there are $5 \cdot 5 \cdot 5 = 125$ of them.

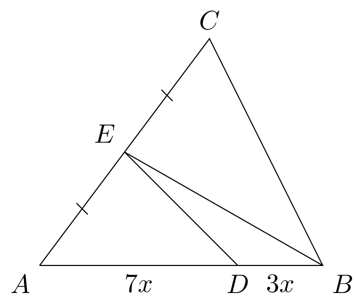
   Alternatively, since the size of the large cube is $30 \times 30 \times 30$, its volume is 30^3. The volume of each small cube is $216 = 6^3$. So there are $\frac{30^3}{6^3} = 5^3 = 125$ small cubes in the large one.

   Answer: (c)

6. (MH 9-10, 2015) Let BE be a median of triangle ABC, and let D be a point on AB such that $\frac{BD}{DA} = \frac{3}{7}$. What is the ratio of the area of triangle BED to that of triangle ABC?

   (a) $\frac{3}{10}$
   (b) $\frac{10}{3}$
   (c) $\frac{3}{20}$
   (d) $\frac{10}{6}$

Solution. If two triangles share a height, then the ratio of their areas is equal to the ratio of their bases. Therefore $\dfrac{A(BDE)}{A(ABE)} = \dfrac{BD}{AB} = \dfrac{3}{10}$ and $\dfrac{A(ABE)}{A(ABC)} = \dfrac{AE}{AC} = \dfrac{1}{2}$. Then $\dfrac{A(BED)}{A(ABC)} = \dfrac{A(BDE)}{A(ABE)} \cdot \dfrac{A(ABE)}{A(ABC)} = \dfrac{3}{10} \cdot \dfrac{1}{2} = \dfrac{3}{20}$.

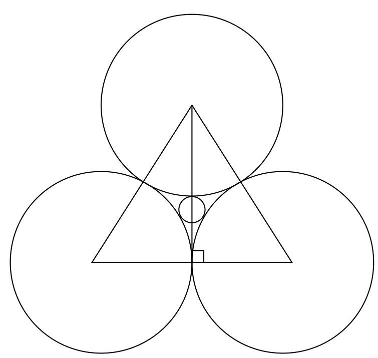


Answer: (c)

7. (MH 9-10, 2015) Three identical coins of radius 1 are placed on a table so that they are mutually tangent. A smaller coin is placed between them tangent to all three. What is the radius of the smaller coin?

   (a) $\frac{1}{3}$
   (b) $\frac{2}{\sqrt{3}} - 1$
   (c) $\sqrt{2} - 1$
   (d) $\frac{1}{2\sqrt{3}}$

   Solution. We are given 3 coins with radius 1. Connecting the centers of the three large coins produces an equilateral triangle with side length 2. Each of its medians has length $\sqrt{3}$, and the point of intersection of the medians divides each median into two parts with ratio $1 : 2$. Thus the center of the triangle is distance $\frac{2}{\sqrt{3}}$ from each of the vertices. This distance is the sum of the radii of a large coin and the small coin, so the small radius is $\frac{2}{\sqrt{3}} - 1$.



   Answer: (b)

8. (MH 9-10, 2010) Two points A and B lie on a sphere of radius 12. The length of the straight line segment joining A and B is $12\sqrt{3}$. What is the length of the shortest path from A to B if every point of the path must lie on the sphere?

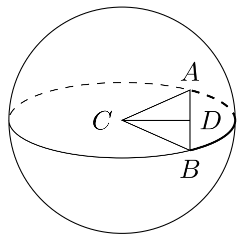
   (a) 6π

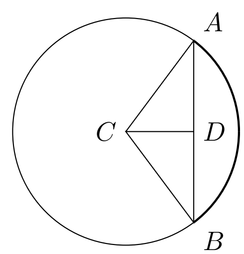
(b) 8π

(c) 9π

(d) 12π

Solution. Let C be the center of the sphere and let D be the midpoint of the line segment AB. Then $|AC| = |BC| = 12$, $|AB| = 12\sqrt{3}$, and $|AD| = |DB| = 6\sqrt{3}$. Therefore $\angle ACD = 60°$, $\angle ACB = 120°$, and the length of the arc AB is $\frac{1}{3}$ of the cicumference, i.e., $\frac{1}{3} \cdot 2\pi \cdot 12 = 8\pi$.





Answer: (b)

9. (MH 9-10, 2015) A paper cone has height 12 inches and the diameter of the base has length 10 inches. The cone is cut along one side and unrolled to form a portion of a disk. What angle of the circle does this portion include?

(a) $\frac{5\pi}{13}$

(b) $\frac{5\pi}{12}$

(c) $\frac{10\pi}{13}$

(d) $\frac{5\pi}{6}$

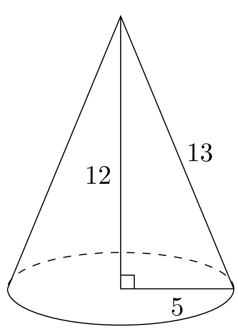
Solution. Since the height is 12 in and the radius of the base is 5 in, using the Pythagorean Theorem we see that the paper disk has radius 13 in. The circumference of the base of the cone is $2\pi r = 10\pi$. This is the length of the circular arc of the portion of the disk that is used to make the cone. The length of the circular arc is proportional to the central angle. For the full central angle, which is 2π, the length is the circumference of the whole circle, which is 26π. Since we want to know for what angle x the arc length is 10π, we set up a proportion. So then we have

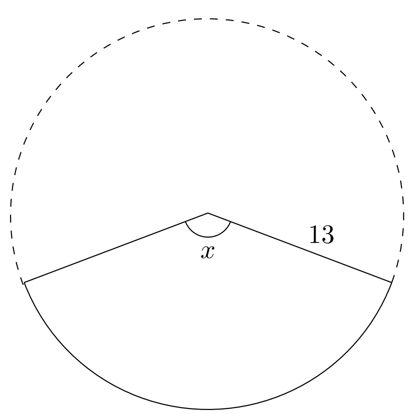
$$\frac{2\pi}{x} = \frac{26\pi}{10\pi}$$

$$20\pi^2 = 26\pi x$$

$$x = \frac{20\pi^2}{26\pi}$$

$$x = \frac{10\pi}{13}.$$





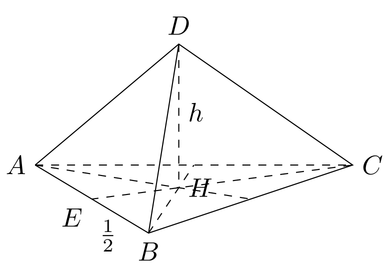
Answer: (c)

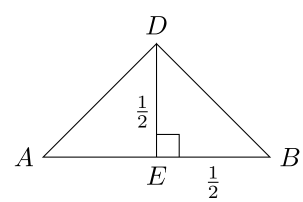
10. (MH 9-10, 2017) Consider a triangular pyramid $ABCD$ with equilateral base ABC of side length 1 such that $AD = BC = CD$ and $m\angle ADB = m\angle BDC = m\angle ADC = 90°$. Find the volume of $ABCD$.

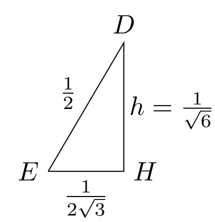
    (a) $\frac{2}{24}$
    (b) $\frac{\sqrt{3}}{24}$
    (c) $\frac{1}{12}$
    (d) $\frac{\sqrt{2}}{24}$

    Solution. First consider the base ABC of the triangular prism $ABCD$. We can construct the medians of the triangle which intersect at the centroid the centroid. Because it is an equilateral triangle, the medians create right angles with the corresponding bases. Now, since the sides are 1 unit in length, the sides are bisected into $\frac{1}{2}$ length pieces. Notice that the medians also form 6 congruent $30° - 60° - 90°$ triangles. Then, we find the length from the triangle side to the centroid to be $\frac{1}{2\sqrt{3}}$. Note that $\triangle ADB$ is an isosceles right triangle. Then, $\triangle BED$ is a $45° - 45° - 90°$ triangle and, therefore, $DE = \frac{1}{2}$. To find the height of the triangular prism, we use the Pythagorean theorem and see that

$$\begin{aligned} h &= \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2\sqrt{3}}\right)^2} \\ &= \sqrt{\frac{3}{12} - \frac{1}{12}} \\ &= \sqrt{\frac{2}{12}} = \frac{1}{\sqrt{6}}. \end{aligned}$$







The area of $\triangle ABC$ is $\frac{\sqrt{3}}{4} \cdot 1^2 = \frac{\sqrt{3}}{4}$. Now, using the volume formula for the pyramid, we have

$$V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{24}.$$

Answer: (d)

Leap Frog

1. (LF 9-10, 2017) A circle is inscribed in the isosceles triangle with respective side lengths 6, 6 and 4. Determine the area of the inscribed circle.

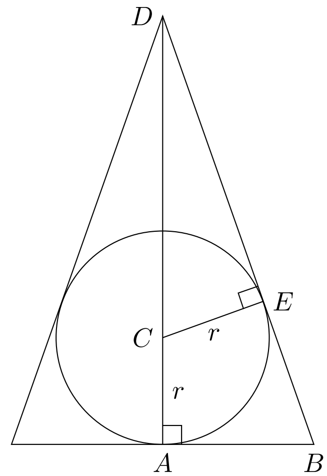
    (a) $\pi/2$
    (b) $3\pi/2$

(c) $5\pi/2$

(d) $7\pi/2$

(e) None of these

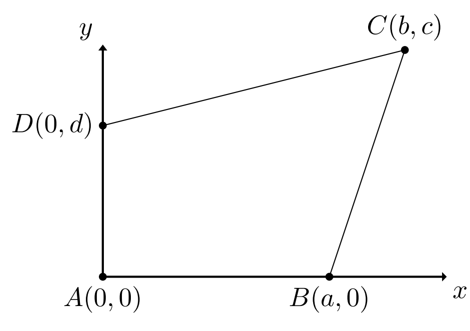
Solution. Label the figure as indicated below.



We have $2 = AB = EB$ so $DE = 4$. Also, by the Pythagorean Theorem applied to the right triangle $\triangle DAB$, we have that $DA = 4\sqrt{2}$. The pair os similar triangles $\triangle DEC \sim \triangle DAB$ implies proportional sides $DA/DE = AB/CE$, that is, $4\sqrt{2}/4 = 2/r$. Therefore $r = \sqrt{2}$. Thus, the area of the circle is $\pi r^2 = 2\pi$, which is none of the answers provided.

Answer: (e)

2. (LF 9-10, 2015) Quadrilateral $ABCD$ in the Cartesian plane is pictured below. Determine the area enclosed by $ABCD$. (You may assume $b > a$ and $c > d$ as pictured.)



(a) Area $= \frac{1}{4}(a+b)(d+c)$

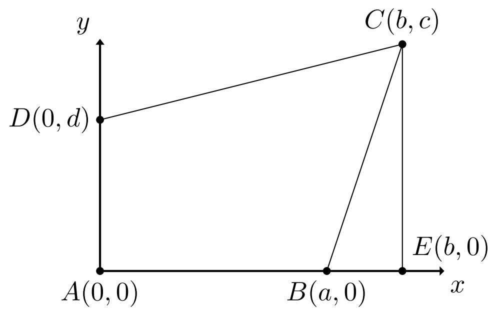
(b) Area $= \frac{1}{4}(a+d)(b+c)$

(c) Area $= \frac{1}{2}(ad+bc)$

(d) Area $= \frac{1}{2}(ac+bd)$

(e) None of these

Solution. Drop a perpendicular segment CE from the point C to the x-axis as pictured below.



The area enclosed by $ABCD$ is the difference of the trapeziod area $AECD$ and the triangle area BEC.

$$\begin{aligned}\text{Area}(ABCD) &= \text{Area}(AECD) - \text{Area}(BEC)\\ &= \frac{1}{2}(c+d)b - \frac{1}{2}(b-a)c\\ &= \frac{1}{2}(ac+bd)\end{aligned}$$

Answer: (d)

3. (LF 9-12, 2005) What is the volume of the cube that circumscribes the sphere that circumscribes the cube that circumscribes the sphere of radius 1 inch?

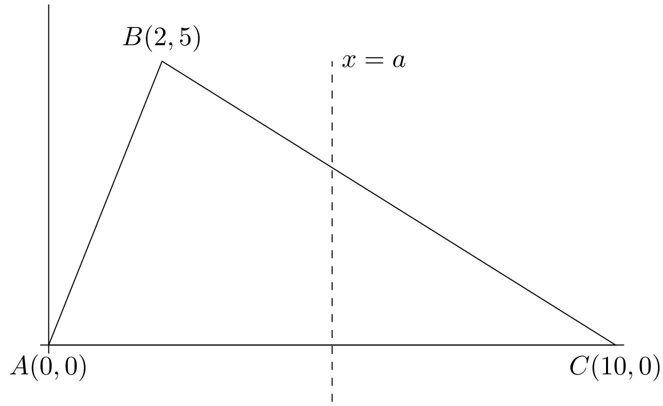
   (a) $9\sqrt{3}$ in^3
   (b) $16\sqrt{2}$ in^3
   (c) $24\sqrt{3}$ in^3
   (d) $54\sqrt{2}$ in^3
   (e) None of these

   Solution 1. Place everything in a 3-dimensional coordinate system where the center of the sphere-cube complex is at the origin $(0,0,0)$. The corner of the smaller cube in the first octant has coordinates $(1,1,1)$, and is $\sqrt{3}$ inches from the origin. So the larger sphere has radius equal to $\sqrt{3}$ inches. This then means that the corner of the larger cube in the first octant is the intersection of the three planes $x=\sqrt{3}$, $y=\sqrt{3}$, $z=\sqrt{3}$, in other words, the point $(\sqrt{3},\sqrt{3},\sqrt{3})$. From this, we see that the side lengths of the larger cube are all equal to $2\sqrt{3}$. The volume of the cube is then $(2\sqrt{3})^3 = 24\sqrt{3}$ in^3.

   Solution 2. Since the diameter of the smaller sphere is 2, the smaller cube is $2\times 2\times 2$. Therefore the diameter of the larger sphere is $2\sqrt{3}$. Thus the larger cube is $2\sqrt{3}\times 2\sqrt{3}\times 2\sqrt{3}$, and its volume is $(2\sqrt{3})^3 = 24\sqrt{3}$ in^3.

   Answer: (c)

4. (LF 9-10, 2015) What is the value of a so that the vertical line $x=a$ divides the triangle $\triangle ABC$ pictured below into two regions of equal area?



(a) $a=\sqrt{7}$

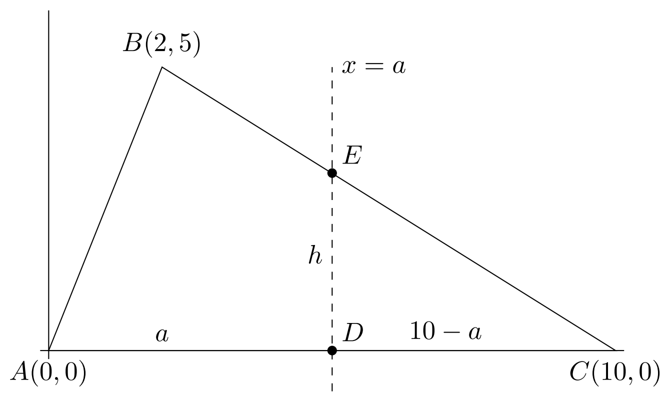
(b) $a=\frac{7}{2}$

(c) $a=3$

(d) $a=10-2\sqrt{10}$

(e) None of these

Solution. First, note that the area of $\triangle ABC$ is 25. Now, label the points D and E as pictured below. Let $h=DE$, the height of $\triangle DCE$



The slope of BC is

$$\text{Slope}(BC)=\frac{5}{2-10}=-\frac{5}{8}.$$

On the other hand,

$$\text{Slope}(EC)=-\frac{h}{10-a},$$

so

$$-\frac{h}{10-a}=-\frac{5}{8}$$

$$h=\frac{5}{8}(10-a).$$

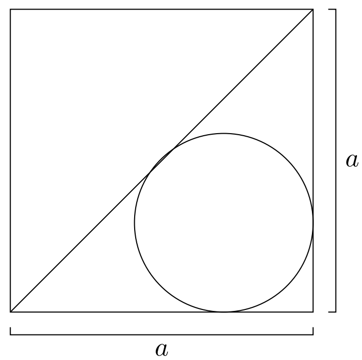
Thus, the area of $\triangle DCE$ is $\frac{1}{2}(10-a)\cdot\frac{5}{8}(10-a)=\frac{5}{16}(10-a)^2$. We want

$$\frac{5}{16}(10-a)^2=\frac{25}{2}.$$

There are two solutions: $a = 10 \pm 2\sqrt{10}$. Since a must be less than 10, $a = 10 - 2\sqrt{10}$ is the answer.

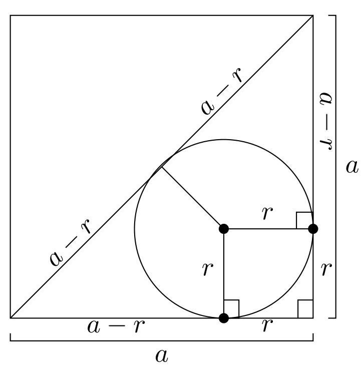
Answer: (d)

5. (LF 9-10, 2015) In the figure below, the rectangle is a square, whose side lengths are all equal to the value a, and the circle is inscribed as pictured. Determine the radius, r, of the inscribed circle.



(a) $r = a(\frac{\sqrt{2}}{2})$
(b) $r = a(1 - \frac{\sqrt{2}}{2})$
(c) $r = a(\sqrt{2} - 1)$
(d) $r = a(2 - \sqrt{2})$
(e) None of these

Solution. Label the figure as follows.

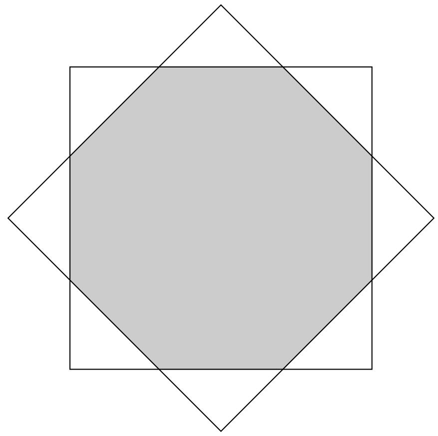


Notice that the diagonal has length $\sqrt{2}a$, so we get the equation $\sqrt{2}a = 2(a - r)$. Solving for r gives us

$$r = a(1 - \sqrt{2}/2)$$

Answer: (b)

6. (LF 9-10, 2015) Two $2' \times 2'$ squares share the same center and one square is rotated $45°$ with respect to the other square (see picture below). Determine the shaded area that is enclosed by both squares.



(a) $4\sqrt{2} - 4$ ft^2

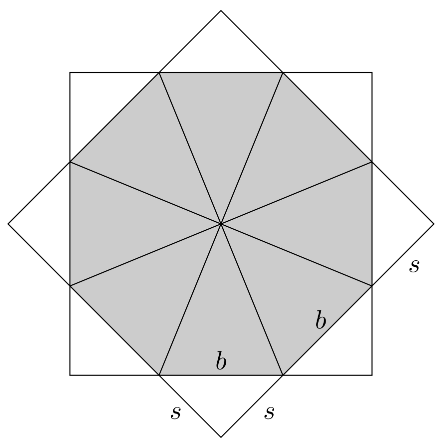
(b) $4\sqrt{2} + 4$ ft^2

(c) $2\sqrt{2} + 2$ ft^2

(d) $8\sqrt{2} - 8$ ft^2

(e) None of these

Solution. We first note that the shaded octagon is a regular octagon due to its rotational symmetry. We can then divide the shaded octagon into 8 isosceles triangles, as pictured below. We have also labeled lengths b and s, as pictured.



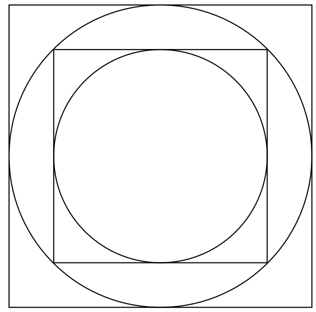
This gives us two equations, $b^2 = 2s^2$ (From the Pythagorean Theorem) and $b + 2s = 2$, since the side length of the square is 2. Substitute $b = \sqrt{2}s$ in the second equation to get $s = 2/(2 + \sqrt{2})$. Substitute this back into the equation $b = \sqrt{2}s$ to get

$$\begin{aligned} b &= \frac{2\sqrt{2}}{2+\sqrt{2}} \\ &= \frac{2\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} \\ &= 2\sqrt{2} - 2 \end{aligned}$$

The area of one of the eight shaded isosceles triangles is then equal to $1/2 \cdot (2\sqrt{2} - 2) \cdot 1 = \sqrt{2} - 1$. Thus the total shaded area is $8\sqrt{2} - 8$.

Answer: (d)

7. (LF 9-10, 2017) A circle is inscribed in a square. A square is inscribed in that circle. A second circle is inscribed in that square. What is the ratio of the area of the smallest circle to the area of the largest square?



(a) $\pi/2$
(b) $\pi^2/4$
(c) $\pi/8$
(d) $\pi^2/16$
(e) None of these

Solution. If a circle inscribed in a square has radius r, then the square has side length $2r$. The respective areas are πr^2 and $4r^2$ and the ratio of the area of the circle to that of the square is $\frac{\pi}{4}$. For a square inscribed in a circle of radius r, the diagonal of the square is a diameter length $2r$ and the side length of the square is $\sqrt{2}r$. The areas are $2r^2$ and πr^2 with a ratio from the square to circle of $\frac{2}{\pi}$. The ratio of the area of the smallest circle to the area of the largest square is the product

$$\frac{\pi}{4} \cdot \frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{\pi}{8}.$$

Answer: (c)

8. (MH 9-10, 2010) A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$.

(a) $\frac{1}{12}$
(b) $\frac{1}{6}$
(c) 6
(d) 12

Solution. The surface area of the cylinder is $2\pi r^2 + 2\pi rh = 12$. Its volume is $\pi r^2 h = 1$. Dividing the first equation by the second, we get $\frac{2}{h} + \frac{2}{r} = 12$, so $\frac{1}{r} + \frac{1}{h} = 6$.

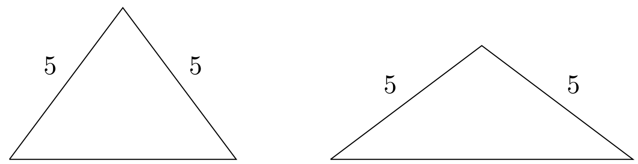
Answer: (c)

More Problems

1. (MH 9-10) An isosceles triangle has equal sides of length 5. Determine the length of the third side.

(a) 4
(b) 3
(c) 6
(d) It cannot be determined from the information given

Solution. Based on the given information we cannot determine the length of the third side.



Answer: (d)

2. (MH 9-10, 2010) An insulated cup is created by taking two cylinders of different radii and heights and filling in the interior with an insulating material, then sealing. If the outer cylinder has height 9 cm and radius 4 cm, and the inner cylinder has height 8 cm and radius 3 cm, determine the volume of insulation.

   (a) 72π cubic centimeters
   (b) 36π cubic centimeters
   (c) 9π cubic centimeters
   (d) 8π cubic centimeters

   Solution. The volume of the outer cylinder is $\pi \cdot 4^2 \cdot 9$, and the volume of the inner cylinder is $\pi \cdot 3^2 \cdot 8$, so the volume of insulation is $\pi \cdot 16 \cdot 9 - \pi \cdot 9 \cdot 8 = \pi \cdot 9 \cdot 8 = 72\pi$ cubic centimeters.

   Answer: (a)

3. (MH 11-12, 2009) If a rectangular box has sides, front and bottom faces with areas of $2x$, $y/2$ and xy in^2 respectively, what is the volume of the solid in cubic inches?

   (a) $2x^2y$
   (b) x^2y^2
   (c) $\frac{xy^2}{2}$
   (d) xy
   (e) None of the above

   Solution. Let l, w, and h denote the length, width, and height of the box, respectively. Then we have $wh = 2x$, $lh = y/2$, and $lw = xy$. The product of these three equations is $whlhlw = 2xy/2 \cdot xy$, or equivalently, $l^2w^2h^2 = x^2y^2$, so the volume of the box is $lwh = xy$.

   Answer: (d)

4. (MH 9-10, 2009) A sphere with radius r has volume A. If the volume of the sphere is doubled, what is the new radius in terms of r?

   (a) $2r$
   (b) $8r$
   (c) $\sqrt[3]{2}r$
   (d) $\frac{r}{\sqrt[3]{2}}$

Solution. Let the new radius be R. The volume of the bigger (new) sphere is $\frac{4}{3}\pi R^3 = 2 \cdot \frac{4}{3}\pi r^3$, so $R^3 = 2r^3$. Therefore $R = \sqrt[3]{2}r$.

Answer: (c)

5. (MH 9-10, 2009) A cube with side length 10 rests inside a sphere so that each of the eight vertices of the cube touch the surface of the sphere. What is the volume of this sphere?

   (a) $\frac{500}{3}\pi$
   (b) $500\sqrt{3}\pi$
   (c) $\frac{4000}{3}\pi$
   (d) $4000\sqrt{3}\pi$

   Solution. Let us place the cube in a coordinate system so that its center is at the origin. Then its vertex in the first octant has coordinates $(5, 5, 5)$. So the radius of the sphere is $r = \sqrt{5^2 + 5^2 + 5^2} = \sqrt{75}$. Therefore its volume is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 75\sqrt{75} = 100\pi\sqrt{75} = 500\sqrt{3}\pi$.

   Answer: (b)

6. (MH 11-12, 2009) Two circles of radius 2 are drawn so that each circle passes through the center of the other. What is the perimeter of the region of overlap?

   (a) $\frac{4\pi}{3}$
   (b) 4π
   (c) 2π
   (d) $\frac{8\pi}{3}$
   (e) None of the above.

   Solution. Notice that the points of intersection of the circles and their centers form equilateral triangles. Thus, the arc measure between the points of intersection is $120°$. Now, we will calculate the corresponding fraction of the circumference of one circle:

   $$\text{Half of desired perimeter} = \frac{120°}{360°}(2\pi r) = \frac{1}{3}(4\pi) = \frac{4\pi}{3}.$$

   Now, doubling this, we get that the perimeter of the region of overlap is $\frac{8\pi}{3}$.

   Answer: (d)