Symmetry of the power sum polynomials

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Outline

• Power sum polynomials
• History
• Recursive definition
• Identity involving Bernoulli numbers
• Symmetry
• Open questions
Recall these familiar formulas from Calculus:

\[
\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n + 1)}{2}
\]

\[
\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
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\[
\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}
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Power sum polynomials

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\[
\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}
\]
**History**

- Pythagoreans (c. 570-500 BCE), Greece

  ![Pythagorean Diagram](image)

- Abu Ali al-Hasan (965-1039), Egypt

  \[
  (n + 1) \sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n} \sum_{i=1}^{p} i^k
  \]

<table>
<thead>
<tr>
<th>$1^k + 2^k + 3^k + \ldots + n^k$</th>
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• Pascal (1623-1662), France

\[ (n + 1)^{p+1} - \left( 1 + n + \binom{p+1}{2} \sum_{k=1}^{n} k^{p-1} \right. \]

\[ + \left( \binom{p+1}{3} \sum_{k=1}^{n} k^{p-2} + \ldots + (p+1) \sum_{k=1}^{n} k \right) \]

\[ = (p + 1)(1^p + 2^p + 3^p + \ldots + n^p) \]

• 1900s

If \( n \) is prime,

\[ 1^p + 2^p + \ldots + n^p \equiv \begin{cases} 
-1 \pmod{n} & \text{if } n - 1 \mid p \\
0 \pmod{n} & \text{if } n - 1 \nmid p 
\end{cases} \]
Definition 1

For $n \in \mathbb{R}$, let $S_1(n) = \frac{n(n + 1)}{2}$.

For $p \geq 2$ and $n \in \mathbb{R}$, we define

$$S_p(n) = \frac{1}{p + 1} \left[ (n + 1)((n + 1)^p - 1) - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i(n) \right].$$
Recursive definition

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**Theorem 2**

For $p, n \in \mathbb{N}$, $S_p(n) = \sum_{k=1}^{n} k^p$. 

Bernoulli numbers, $B_m$, are defined as follows:

**Definition 3**

Let $B_0 = 1$, and for each $m \geq 1$,

$$
\sum_{i=0}^{m} \binom{m+1}{i} B_i = 0.
$$

The first few Bernoulli numbers are:

$$1, \ -\frac{1}{2}, \ \frac{1}{6}, \ 0, \ -\frac{1}{30}, \ 0, \ \frac{1}{42}, \ \ldots$$

Note that for $m \geq 3$ odd, $B_m = 0$. 
Identity involving Bernoulli numbers

**Theorem 4**

For $m, k \in \mathbb{Z}$, $m \geq 1$, $0 \leq k \leq m$,

$$(-1)^{m-k} \binom{m}{k} B_{m-k} = \sum_{i=k}^{m} \binom{m}{i} \binom{i}{k} B_{m-i}$$
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Sketch of Proof

Induction on both \( m \) and \( k \)

- Consider the case \((m, k) = (m, m)\)
- Consider the case \((m, k) = (m, 0)\)
- Assume the statement holds for \((m, k)\) and show it holds for \((m + 1, k + 1)\)
Identity involving Bernoulli numbers

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Theorem 5

For each \( p \in \mathbb{N} \), \( S_p(n) \) has symmetry about \(-\frac{1}{2}\). Namely, it is symmetric about the vertical line at \(-\frac{1}{2}\) if \( p \) is odd, and symmetric about the point \((-\frac{1}{2}, 0)\) if \( p \) is even.

\[
S_p(n) = 1^{p+1} \sum_{i=0}^{p} (-1)^i \binom{p+1}{i} n^{p+1-i} - i
\]

Expanding \( S_p(-n) \) using the binomial theorem, combining like terms, and using the previous identity for Bernoulli numbers yields

\[
S_p(-n) =
\begin{cases} 
  S_p(n) & \text{if } p \text{ is odd}, \\
  -S_p(n) & \text{if } p \text{ is even}.
\end{cases}
\]

Corollary 6

For each \( p \in \mathbb{N} \), the roots of \( S_p(n) \) are symmetric about \(-\frac{1}{2}\). When \( p \) is even, \( S_p(n) \) has \(-\frac{1}{2}\) as a root.
Theorem 5

For each $p \in \mathbb{N}$, $S_p(n)$ has symmetry about $-\frac{1}{2}$. Namely, it is symmetric about the vertical line at $-\frac{1}{2}$ if $p$ is odd, and symmetric about the point $(-\frac{1}{2}, 0)$ if $p$ is even.

Sketch of Proof

Faulhaber’s (Bernoulli’s) Formula

$$S_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} (-1)^i \binom{p+1}{i} B_i n^{p+1-i}$$

Expanding $S_p(-(n+1))$ using the binomial theorem, combining like terms, and using the previous identity for Bernoulli numbers yields

$$S_p(-(n+1)) = \begin{cases} S_p(n) & \text{if } p \text{ is odd}, \\ -S_p(n) & \text{if } p \text{ is even}. \end{cases}$$
Theorem 5

For each $p \in \mathbb{N}$, $S_p(n)$ has symmetry about $-\frac{1}{2}$. Namely, it is symmetric about the vertical line at $-\frac{1}{2}$ if $p$ is odd, and symmetric about the point $(-\frac{1}{2}, 0)$ if $p$ is even.

Sketch of Proof

Faulhaber's (Bernoulli's) Formula

$$S_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} (-1)^i (p+1)^{i+1} B_i n^{p+1-i}$$

Expanding $S_p(-n-1)$ using the binomial theorem, combining like terms, and using the previous identity for Bernoulli numbers yields

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Corollary 6

For each $p \in \mathbb{N}$, the roots of $S_p(n)$ are symmetric about $-\frac{1}{2}$. When $p$ is even, $S_p(n)$ has $-\frac{1}{2}$ as a root.
Open questions

- How many (distinct) real roots does $S_p(n)$ have?
- Where are the real roots located?
- Where are the complex roots located?
Thank you!