Triangle Centers

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(based on joint work with Larry Cusick)

Junior Seminar in Pure Mathematics
California State University, Fresno
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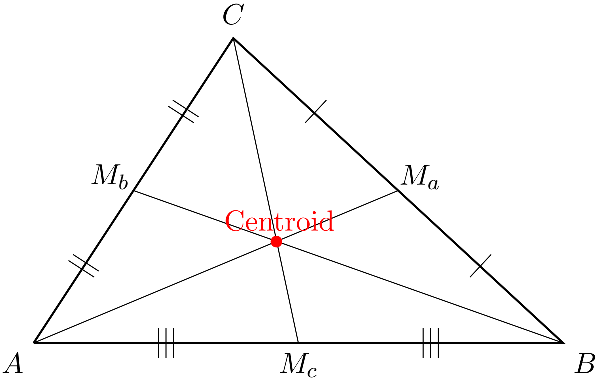
Outline

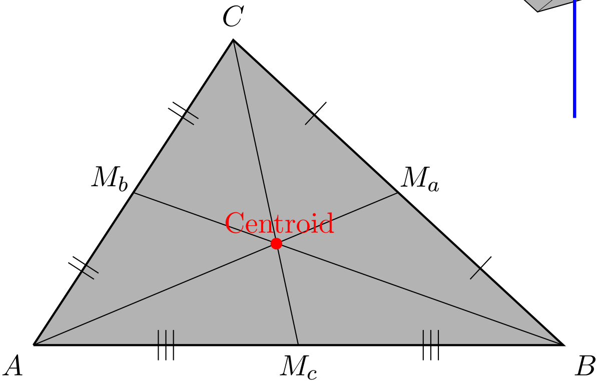
- Triangle Centers
  - Well-known centers
    - Center of mass
    - Incenter
    - Circumcenter
    - Orthocenter
  - Not so well-known centers (and Morley's theorem)
  - More recently discovered centers
- Better coordinate systems
  - Trilinear coordinates
  - Barycentric coordinates
  - So what qualifies as a triangle center?
- Open problems (= possible projects)

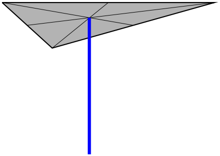
Centroid (center of mass)



Three medians in every triangle are concurrent.
Centroid is the point of intersection of the three medians.

Centroid (center of mass)





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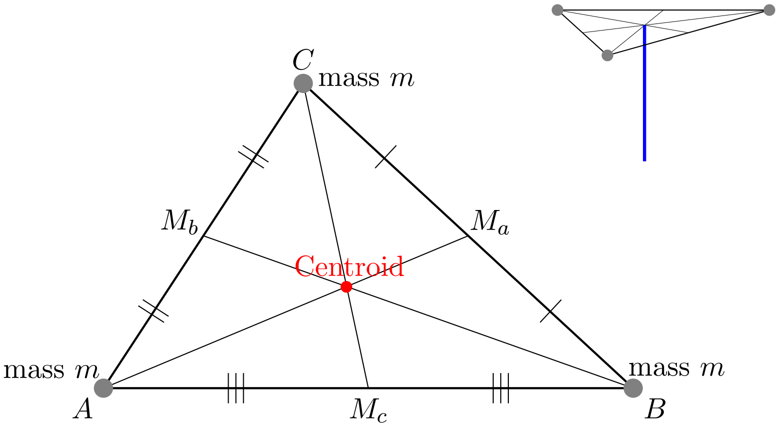
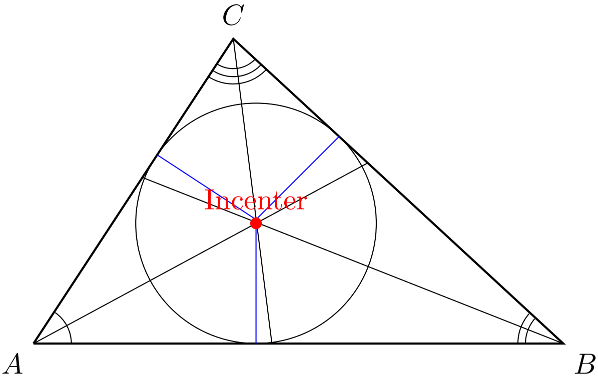


Three medians in every triangle are concurrent.

Centroid is the point of intersection of the three medians.

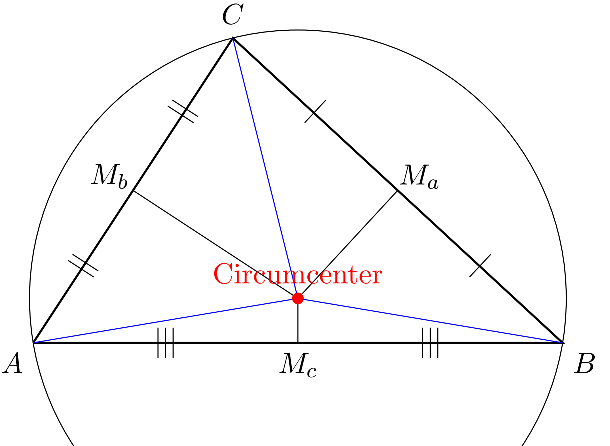
Incenter



Three angle bisectors in every triangle are concurrent.

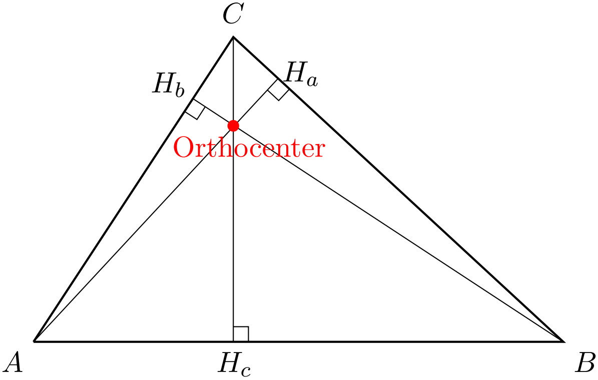
Incenter is the point of intersection of the three angle bisectors.

Circumcenter



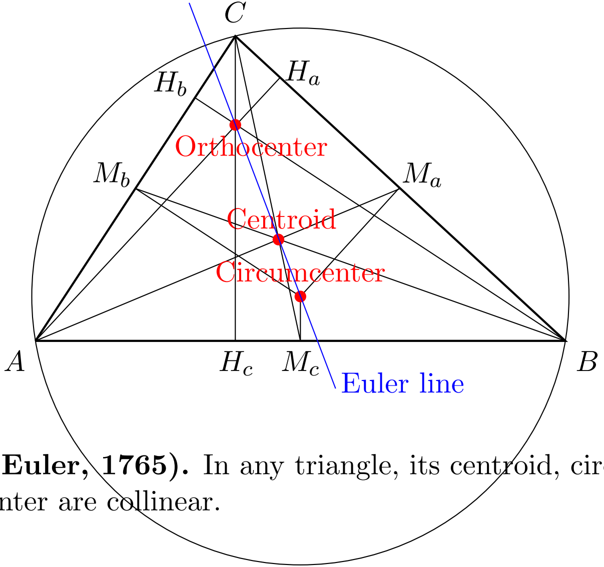
Three side perpendicular bisectors in every triangle are concurrent. Circumcenter is the point of intersection of the three side perpendicular bisectors.

Orthocenter



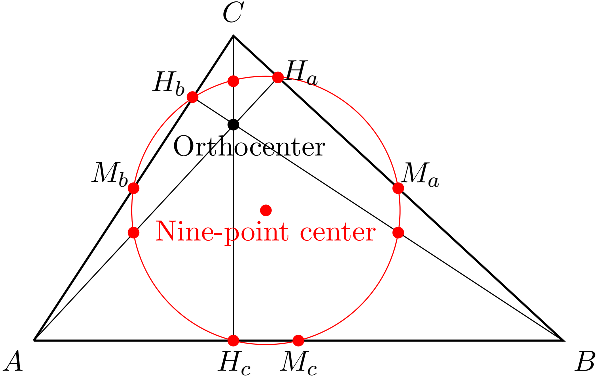
Three altitudes in every triangle are concurrent.
Orthocenter is the point of intersection of the three altitudes.

Euler Line



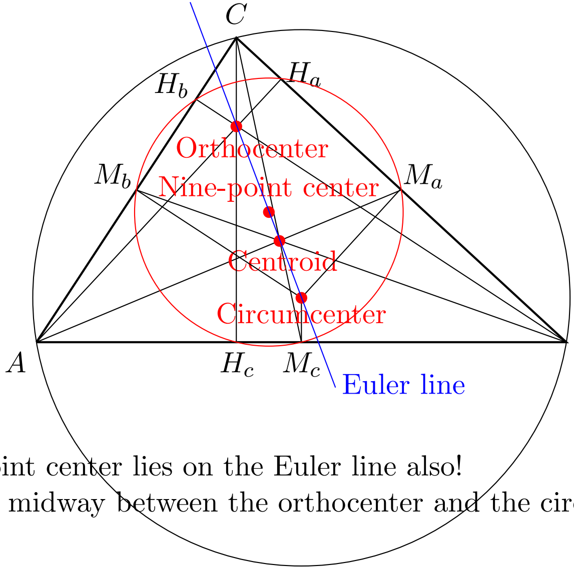
Theorem (Euler, 1765). In any triangle, its centroid, circumcenter, and orthocenter are collinear.

Nine-point circle



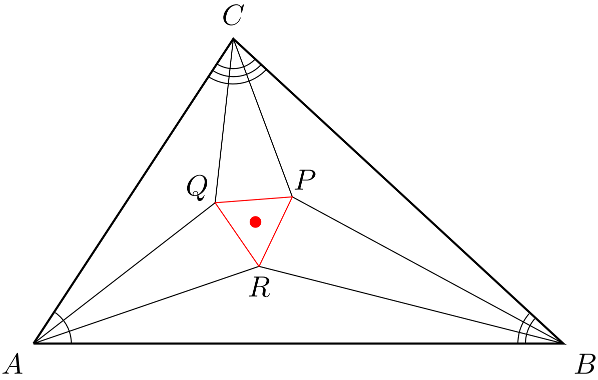
The midpoints of sides, feet of altitudes, and midpoints of the line segments joining vertices with the orthocenter lie on a circle. Nine-point center is the center of this circle.

Euler Line



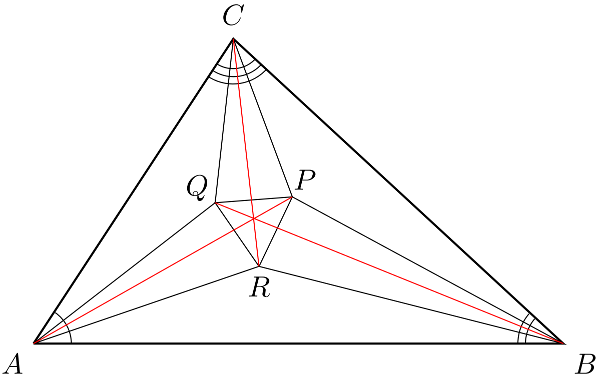
The nine-point center lies on the Euler line also!
It is exactly midway between the orthocenter and the circumcenter.

Morley's Theorem



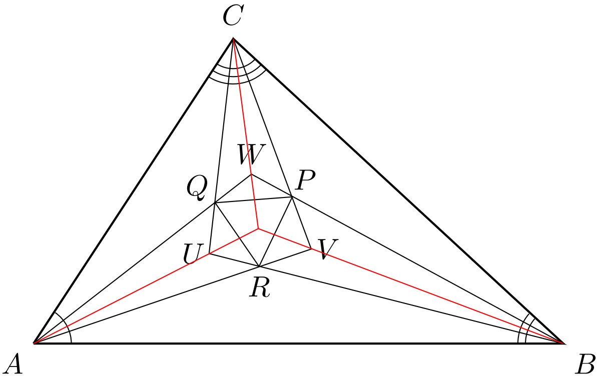
Theorem (Morley, 1899). $\triangle PQR$ is equilateral.
The centroid of $\triangle PQR$ is called the first Morley center of $\triangle ABC$.

Classical concurrencies



The following line segments are concurrent:
AP, BQ, CR

Classical concurrencies



The following line segments are concurrent:

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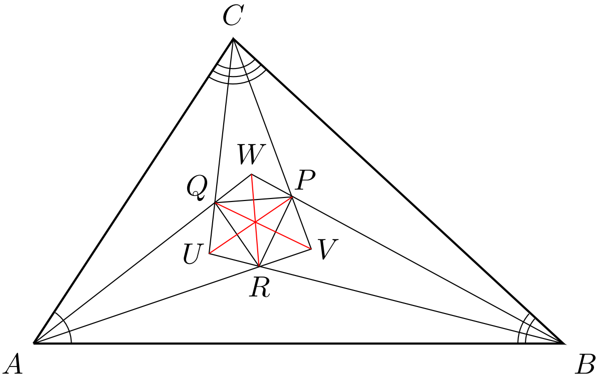
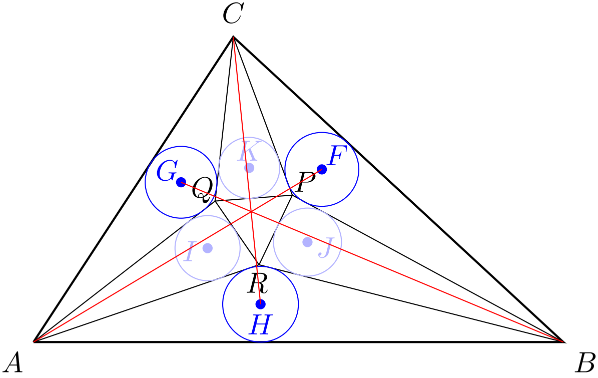
Classical concurrencies



The following line segments are concurrent:

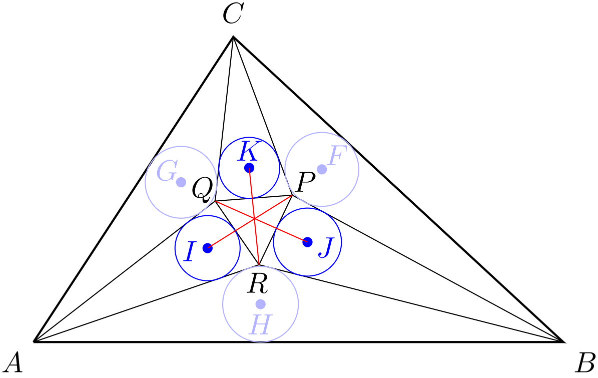
AP, BQ, CR AU, BV, CW PU, QV, RW

New Concurrency I



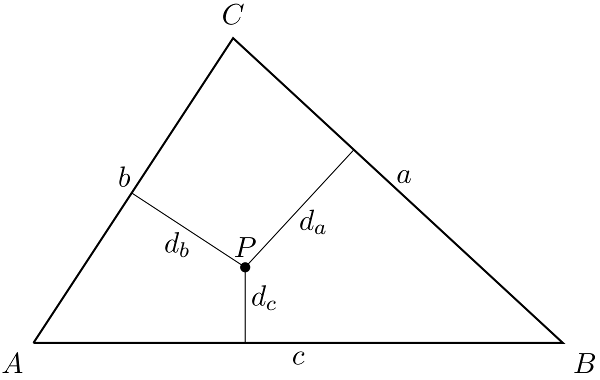
Theorem (Cusick and Nogin). The following line segments are concurrent:
AF, BG, CH

New Concurrency II



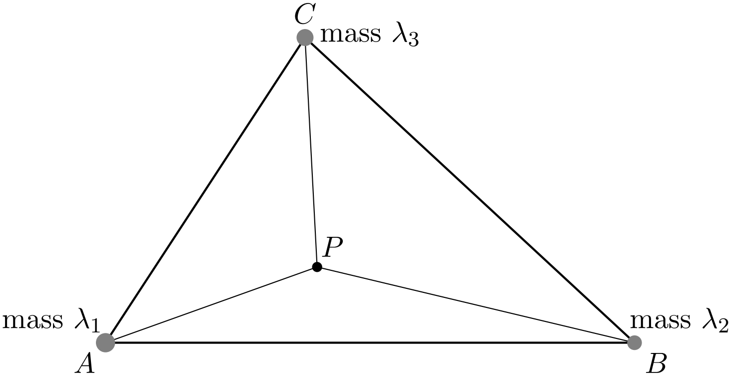
Theorem (Cusick and Nogin). The following line segments are concurrent:
AF, BG, CH PI, QJ, RK

Trilinear Coordinates



Trilinear coordinates: triple (t_1, t_2, t_3) such that $t_1 : t_2 : t_3 = d_a : d_b : d_c$
e.g. $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$

Barycentric Coordinates

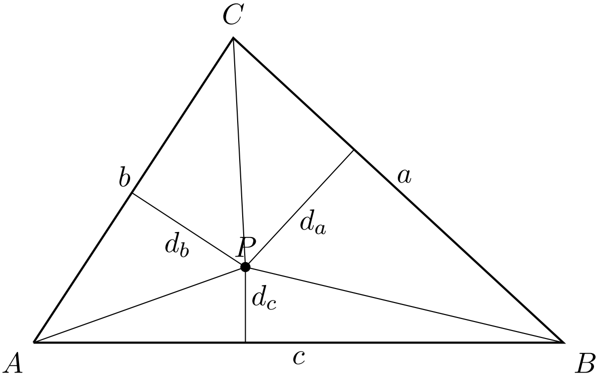


Barycentric coordinates: triple $(\lambda_1, \lambda_2, \lambda_3)$ such that P is the center of mass of the system {mass λ_1 at A, mass λ_2 at B, mass λ_3 at C}, i.e.

$\lambda_1 \overrightarrow{A} + \lambda_2 \overrightarrow{B} + \lambda_3 \overrightarrow{C} = (\lambda_1 + \lambda_2 + \lambda_3) \overrightarrow{P}$

$\lambda_1 : \lambda_2 : \lambda_3 = \text{Area}(PBC) : \text{Area}(PAC) : \text{Area}(PAB)$

Trilinears vs. Barycentrics



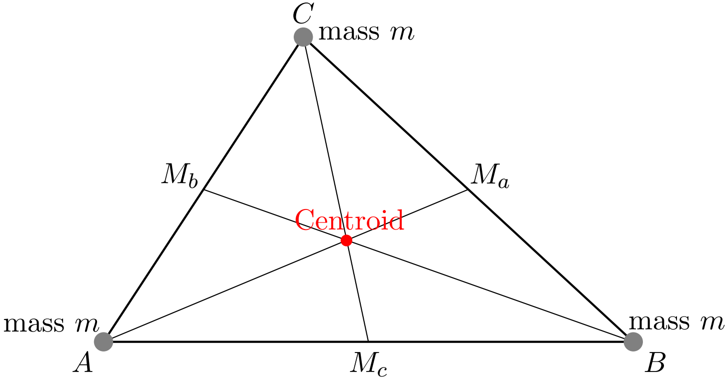
Trilinears: $t_1 : t_2 : t_3 = d_a : d_b : d_c$

Barycentrics: $\lambda_1 : \lambda_2 : \lambda_3 = \text{Area}(PBC) : \text{Area}(PAC) : \text{Area}(PAB)$

$$= ad_a : bd_b : cd_c$$

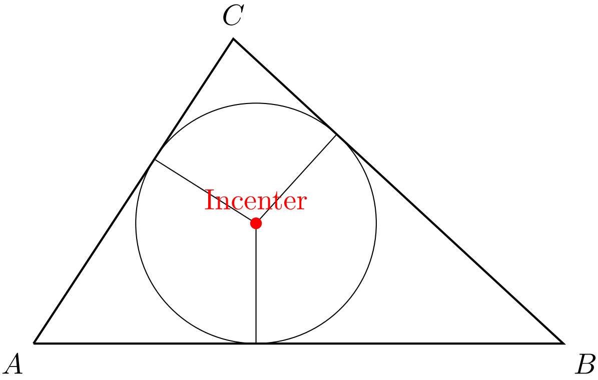
$$= at_1 : bt_2 : ct_3$$

Centroid (center of mass)



Trilinear coordinates: $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$
Barycentric coordinates: $1 : 1 : 1$

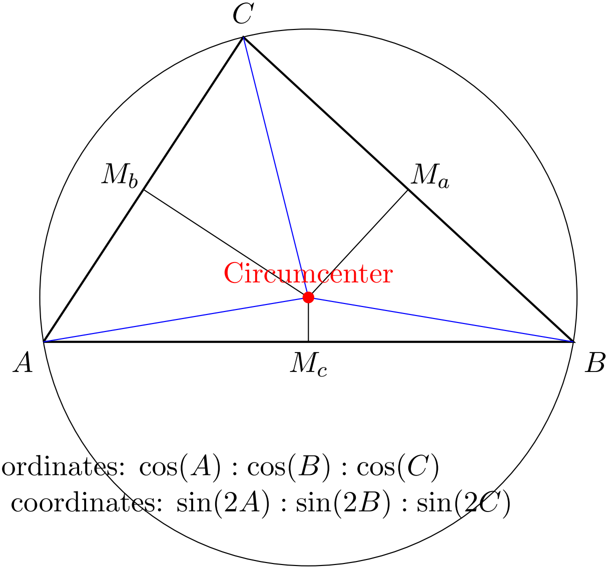
Incenter



Trilinear coordinates: $1 : 1 : 1$

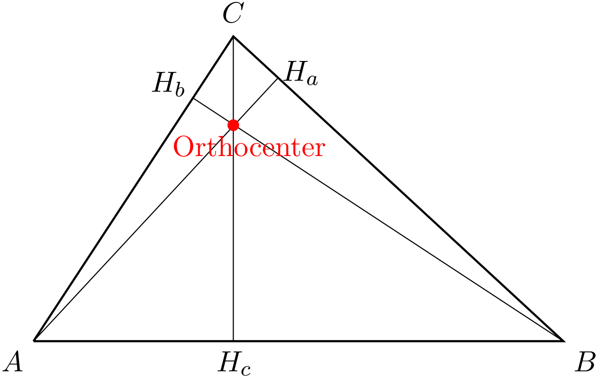
Barycentric coordinates: $a : b : c$

Circumcenter



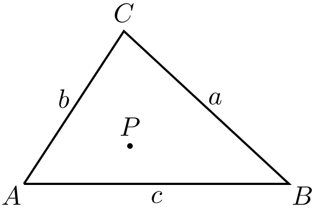
Trilinear coordinates: $\cos(A) : \cos(B) : \cos(C)$
Barycentric coordinates: $\sin(2A) : \sin(2B) : \sin(2C)$

Orthocenter



Trilinear coordinates: $\sec(A) : \sec(B) : \sec(C)$
Barycentric coordinates: $\tan(A) : \tan(B) : \tan(C)$

What is a triangle center?



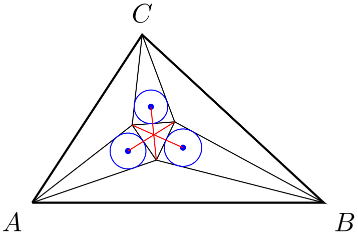
A point P is a triangle center if it has a trilinear representation of the form

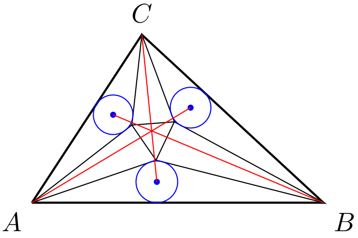
$$f(a,b,c) : f(b,c,a) : f(c,a,b)$$

such that $f(a,b,c) = f(a,c,b)$
(such coordinates are called homogeneous in the variables a, b, c).

Open problems (= possible projects)

1. Find the trilinear or barycentric coordinates of both points of concurrency:

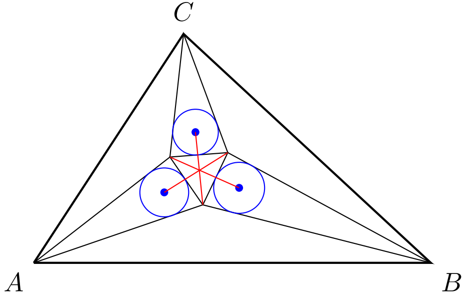




2. Are these points same as some known triangle centers?

Open problems (= possible projects)

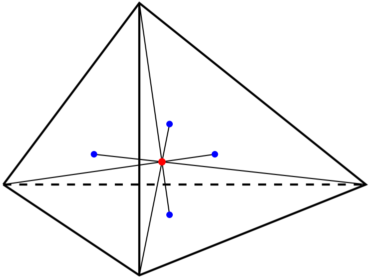
3. Find an elementary geometry proof of this concurrency:

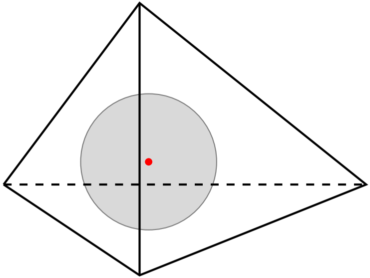


4. Any other concurrencies?

Open problems (= possible projects)

5. Which of the known triangle centers can be generalized to 3D and/or higher dimensions?





Thank you!