Dynamic topological logic of the real line

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Subset interpretation

Let $X$ be a set.

Logical connectives are interpreted as operations on subsets of $X$:

- conjunction $\land$ – as intersection $\cap$
- disjunction $\lor$ – as union $\cup$
- negation $\neg$ – as complement $\overline{\phantom{P}}$
- $(P \to Q) \equiv ((\neg P) \lor Q)$

Given a mapping from propositional variables $(P, Q, \text{etc.})$ to subsets of $X$, every formula is mapped to a subset $X$.

e.g. $P \land Q \mapsto P \cap Q$

$P \lor \neg P \mapsto P \cup \overline{P}$

Definition. Formulas that are always mapped to the whole set $X$ are called valid with respect to interpretation in $X$. 

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Soundness and completeness

Let $X$ be a set.

1. All tautologies (= formulas derivable from axioms) of the classical logic are valid with respect to interpretation in $X$. The classical logic is sound with respect to this interpretation.

2. If $X$ is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in $X$. The classical logic is complete with respect to this interpretation.

The language of classical logic does not distinguish different non-empty sets $X$.
**S4:** $\land, \lor, \neg, \rightarrow, \leftrightarrow, \square$

- Axioms of classical logic
- $\square P \rightarrow P$
- $\square P \rightarrow \square \square P$
- $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$

Rules of inference:

1. $\begin{array}{c} P, P \rightarrow Q \\ \hline Q \end{array}$

2. $\begin{array}{c} P \\ \hline \square P \end{array}$

**Topological interpretation of $\square$:**

$\square P \rightarrow \text{interior}(P)$

**Theorem.** Let $X$ be a topological space. Then $S4$ is sound with respect to interpretation in $X$. 
Completeness

**Theorem.** S4 is complete with respect to all interpretations in all topological spaces $X$, i.e. for any formula $F$, the following statements are equivalent:

1. $F$ is derivable in S4
2. $F$ is valid in each interpretation (for each topological space $X$)
3. $F$ is valid in each interpretation for each $\mathbb{R}^n$
4. $F$ is valid in each interpretation for some $\mathbb{R}^n$

**Corollary.** The modal logic (with operations $\land$, $\lor$, $\neg$, $\rightarrow$, $\square$) does not distinguish $\mathbb{R}^n$’s for different $n$. 
**Dynamic topological systems**

**Definition.** A dynamic topological system is a topological space $X$ with a continuous function $f: X \to X$.

**New modal operator $\odot$:**

$\odot P$ is interpreted as $f^{-1}(P)$. 
S4C

- Axioms of classical logic
  - $\square P \rightarrow P$
  - $\square P \rightarrow \square \square P$
  - $\square(P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$
  - $\circ(P \rightarrow Q) \rightarrow (\circ P \rightarrow \circ Q)$
  - $(\circ \neg P) \leftrightarrow (\neg \circ P)$
  - $(\circ \square P) \leftrightarrow (\square \circ \square P)$

Rules of inference:

1. $\frac{P, P \rightarrow Q}{Q}$

2. $\frac{P}{\square P}$

3. $\frac{P}{\circ P}$
Completeness

**Theorem.** Let $F$ be a formula. The following are equivalent:

1. $F$ is derivable in S4C
2. $F$ is valid with respect to every interpretation in every $\mathbb{R}^n$

However, the above statements are not equivalent to

3. $F$ is valid with respect to every interpretation in $\mathbb{R}$

**Corollary.** The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$. 
Logic of $\mathbb{R}$

**Open question.** Which formulas are valid with respect to interpretation in $\mathbb{R}$?

**New axioms:**

1. $\circ Q \land \Diamond (\circ \neg Q \land \circ \Diamond \neg P \land \Box \Box P) \rightarrow \Diamond (\circ \neg Q \land \Diamond \circ \neg P \land \Diamond \Box \Box P)$

2. $\circ \neg P \land \circ \neg Q \land \Diamond \Box \Box P \land \Diamond \circ (\neg P \land Q) \land \Box \Box S \rightarrow \Diamond (\Diamond \Box \Box P \land \Diamond \circ \neg P \land \circ \Box S)$

(Where $\Diamond = \neg \Box \neg$)

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