Optimizing Prisms of All Shapes and Dimensions

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Perhaps you have solved the following optimization problem in your Calculus I course: Determine the dimensions of a square prism that has a given volume and the smallest possible surface area (to minimize the amount of material needed to manufacture such a box). It is well known that the answer to this problem, or "the optimal box," so to say, is a cube.

This classical problem is generalized in [**4**] where it is proved that, of all prisms of a given volume and with base being a regular polygon with a given number of sides (such as six in Figure 1), the one with the smallest possible surface area satisfies the relationship $h = 2a$ where h is the height of the prism and a is the apothem of the base (the apothem of a regular polygon is the distance from the center of the polygon to any of its sides; the quantity is also called the inradius).

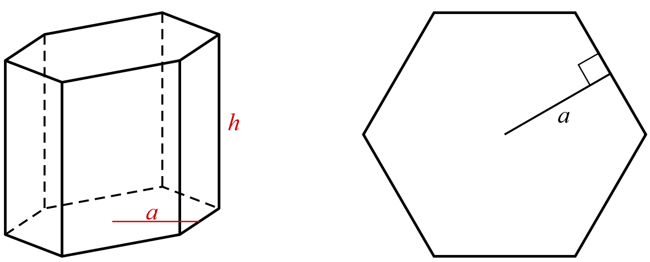

Figure 1. A hexagonal prism with height and apothem shown.

A natural question arises: What about irregular prisms? Suppose we want a prism to have a certain volume, its base to have a certain shape, i.e., be similar to a given irregular polygon (such as shown in Figure 2), or even any given region, and have the smallest possible surface area. Can we say anything about its height, perhaps find some relationship between the height and the size of the base of such an optimal prism?

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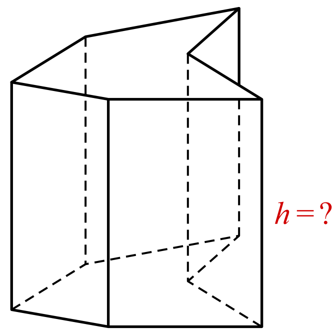

Figure 2. A hexagonal prism with with an irregular base.

Irregular prisms

In this section, we define the apothem for any polygon or, even more generally, for any 2-dimensional figure with a finite boundary and prove that the same relationship $h = 2a$ will hold for the optimal prism with the base similar to the given figure.

Definition. For any connected 2-dimensional region of area A bounded by a curve of finite length P (for perimeter), its apothem is $a = 2A/P$.

For any regular polygon, a is its apothem in the standard sense, or inradius. Also, this generalized notion of apothem for a circle is its radius.

For any convex polygon with k sides $s_1, \dots, s_k$ and distances from any inside point to these sides or their extensions $d_1, \dots, d_k$, respectively, as shown in Figure 3, we have

$$A = \frac{1}{2}(s_1 d_1 + \ldots + s_k d_k),$$

$$a = \frac{2A}{P} = \frac{s_1 d_1 + \ldots + s_k d_k}{s_1 + \ldots + s_k} = \sum_{i=1}^{k} \frac{s_i}{s_1 + \ldots + s_k} d_i,$$

that is, the apothem is a weighted average of those k individual distances.

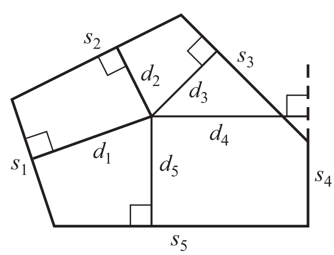

Figure 3. Computing the apothem of an irregular polygon.

If the polygon is not convex, then the above formula remains true if we define d_i to be negative of the distance from the point to the side (or its extension) when the line segment from the point to the side/extension connects “from the outside” of the polygon, as shown in Figure 4.

Also note that if two figures are similar with a scale factor of s, then their apothems have a ratio of s.

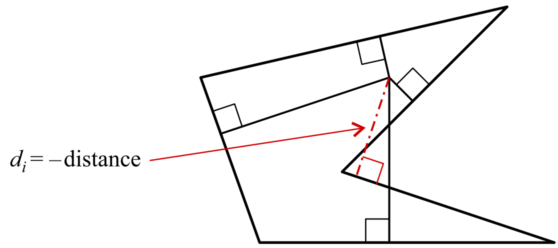

Figure 4. An example of negative d_i for a nonconvex polygon.

Theorem 1. *Of all prisms with volume V and base similar to a given region, the one with $h = 2a$ has the smallest possible surface area, where h is the height of the prism and a is the apothem of the base.*

Proof. Suppose the given region has area A_0 and perimeter P_0, then its apothem is $a_0 = 2A_0/P_0$. For any similar region with apothem a, its area is

$$A(a) = \left(\frac{a}{a_0}\right)^2 A_0 = ca^2$$

where $c = A_0/a_0^2$ and its perimeter is

$$P(a) = \frac{a}{a_0} P_0 = \frac{a}{a_0}\frac{2A_0}{a_0} = 2ca.$$

Observe that $P(a)$ is the derivative of $A(a)$, just as for regular polygons (proved in [**2**] and, in a more general way, in [**1**]). We will show that it follows from [**4**] that the problem of optimizing any such prism is equivalent to the problem of optimizing a square prism. Indeed, the volume of the prism here is

$$V = A(a)h = ca^2h = \frac{c}{4}(4a^2h),$$

i.e., is $c/4$ times larger than that of the square prism with height h and apothem of the base a, and the surface area of our prism is

$$S = 2A(a) + P(a)h = 2ca^2 + 2cah = \frac{c}{4}(8a^2 + 8ah),$$

i.e., $c/4$ times that of the square prism with height h and apothem of the base a. The problem is now equivalent to minimizing $8a^2 + 8ah$ given $4a^2h$. Thus, the optimal prism with a base of any shape must obey the same relationship as that with square base, i.e., $h = 2a$. ■

Observe that, for any such optimal prism, the total area of the top and bottom is

$$2ca^2 = \frac{1}{3}(2ca^2 + 4ca^2) = \frac{1}{3}(2ca^2 + 2cah) = \frac{1}{3}S,$$

one-third of the total surface area. (For the optimal square prism, this is very easy to see: The top and the bottom consist of the two out of six equal faces of a cube.)

Notice that for the analogous 2-dimensional problem (e.g., [**5**, p. 135, problem 1]), the optimal rectangle is a square with the total length of the top and bottom being 1/2 of the total perimeter. The following question naturally arises: Is it a coincidence that we get the fraction $1/2$ in the 2-dimensional optimization problem and the fraction $1/3$ in the 3-dimensional one? In the following section, we will show that this is not at all a coincidence.

Taking optimization to higher dimensions

Let us first consider a special case, the prism with a cube base.

Theorem 2. *Of all n-dimensional prisms with volume V and an $(n-1)$-dimensional cube as its "base," the cube has the smallest possible surface area.*

This is a special case of [**3**, Thm. 2.6a]; we give a proof here as it is useful below.

Proof. Let x be the side of the base so that the height is $h = V/x^{n-1}$. An $(n-1)$-dimensional cube has $2(n-1)$ faces (sometimes called facets), so the surface area of the prism is

$$S = 2x^{n-1} + 2(n-1)x^{n-2}h = 2x^{n-1} + 2(n-1)\frac{V}{x}.$$

Looking for critical points, set

$$S' = 2(n-1)x^{n-2} - 2(n-1)\frac{V}{x^2} = 0$$

which yields $x^n = V$. It is easy to verify that $x = \sqrt[n]{V}$ is an absolute minimum and $h = x$. ∎

Now we consider the general problem: Given any connected $(n-1)$-dimensional solid with a finite surface area, we want to find the n-dimensional prism whose volume is V, base is similar to the given solid, and surface area is minimal.

Since the given $(n-1)$-dimensional solid is similar to the base of the optimized prism, its volume plays the same role as the area of the base in the 3-dimensional prism case, and its surface area plays the same role as the perimeter of the base in the 3-dimensional prism case. Therefore, we denote these measures A and P, respectively, to help show the analogy.

Definition. For any connected m-dimensional solid with volume A and finite surface area P, its apothem is $a = mA/P$.

This arbitrary dimension apothem shares many of the characteristics mentioned for the generalized apothem above: For a regular polytope, a is its inradius. For an m-sphere, $a = mA/P$ is its radius. The only variation is that, for a convex polytope with k faces of areas $s_1, \dots, s_k$ and distances from any inside point to these faces or their extensions $d_1, \dots, d_k$ respectively, we have

$$A = \frac{1}{m}(s_1 d_1 + \dots + s_k d_k),$$

but the weighted average expression for a is the same. Making d_i negative for nonconvex polytopes works analogously, and the remark regarding similarity still holds.

Theorem 3. *Of all n-dimensional prisms with volume V and base similar to a given $(n-1)$-dimensional solid, the one with $h = 2a$ has the smallest possible surface area, where h is the height of the prism and a is the apothem of the base.*

Proof. The proof runs parallel to the proof of Theorem 1: The $(n-1)$-dimensional solid with volume A_0 and surface area P_0 has apothem $a_0 = (n-1)A_0/P_0$. A similar solid with apothem a has volume and surface area, respectively,

$$A(a) = ca^{n-1}, \qquad P(a) = (n-1)ca^{n-2}$$

where here $c = A_0/a_0^{n-1}$. These are again related by $P(a) = A'(a)$. The volume and surface area, respectively, are

$$\begin{aligned} V &= A(a)h = ca^{n-1}h = \frac{c}{2^{n-1}}(2a)^{n-1}h, \\ S &= 2A(a) + P(a)h = 2ca^{n-1} + (n-1)ca^{n-2}h \\ &= \frac{c}{2^{n-1}}\big(2(2a)^{n-1} + 2(n-1)(2a)^{n-2}h\big), \end{aligned}$$

both the constant $c/2^{n-1}$ times the corresponding parameters of a cube prism. By Theorem 2, then, the optimal prism with a base of any shape has the same relationship as that with the cube base, namely $h = 2a$. ■

It follows that, in any optimal n-dimensional prism, the total surface area of the top and bottom must be $1/n$ of the total surface area.

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Summary. We generalize the standard calculus problem of finding the optimal shape of a square prism to allow bases of any shape and then extend to arbitrary dimension. In all cases, the prism with the smallest possible surface area given the volume has height twice the apothem of the base (where the definition of apothem has been suitably generalized).

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