Hoover High School Math League

March 18-19, 2009

Bases other than 10: Solutions

Integers

1. (MH 9-10 2005) Convert 346_{seven} to a base 10 value.
   (a) 181
   (b) 346
   (c) 567
   (d) none of the above

   Solution. $346_{\text{seven}} = 3 \cdot 7^2 + 4 \cdot 7^1 + 6 \cdot 7^0 = 3 \cdot 49 + 4 \cdot 7 + 6 = 181.$

2. (MH 9-10 2006) Convert 128_{16} to a base 10 number.
   (a) 4736
   (b) 200
   (c) 256
   (d) 296

   Solution. $128_{16} = 1 \cdot 16^2 + 2 \cdot 16 + 8 = 296.$

3. (MH 9-10 2005) Convert 432 (base 10) to a base 5 value.
   (a) 3212_{five}
   (b) 2312_{five}
   (c) 432_{five}
   (d) none of the above

   Solution. $432 = 3 \cdot 125 + 2 \cdot 25 + 1 \cdot 5 + 2 = 3212_{\text{five}}.$

4. (MH 9-10 2006) Convert 384 (base 10) to a hexadecimal (base 16) number.
   (a) 100_{16}
   (b) 120_{16}
   (c) 140_{16}
   (d) 180_{16}

   Solution. $384 = 1 \cdot 16^2 + 8 \cdot 16 + 0 = 180_{16}.$

5. (MH 9-10 2008) Which of the following represents the number 34 (base 10) as a base-6 number?
   (a) 100_6
   (b) 54_6
   (c) 34_6
   (d) None of the above

   Solution. $34 = 5 \cdot 6 + 4 = 54_6.$

6. (MH 9-10 1998) $43_{nine} =$
   (a) 123_{five}
   (b) 125_{five}
   (c) 234_{five}
   (d) 124_{five}

   Solution. $43_{nine} = 4 \cdot 9 + 3 = 39 = 1 \cdot 25 + 2 \cdot 5 + 4 = 124_{five}$.

7. (MH 11-12 2005) The binary system uses base-2 numbers (i.e., the only allowable digits are 0 and 1). Which of the following base 2 numbers is divisible by 2?
   (a) 111
   (b) 110
   (c) 101
   (d) 011
   (e) All of the above are divisible by 2.

   Solution. Since each power of 2 except 2^0 is divisible by 2, a number is divisible by 2 if and only if its base 2 representation ends with 0.
   Note: this is the base 2 analogue of the fact that a number (written in base 10) is divisible by 10 if and only if it ends with 0.

8. (MH 11-12 2005) In the binary number system, what is 101 plus 110?
   (a) 211
   (b) 111
   (c) 1111
   (d) 1011
   (e) None of the above

   Solution. Addition in base b is done similarly to addition in base 10: we add units digits first, then ”tens” digits, etc., and carry over whenever we get a sum larger than b, so $101 + 110 = 1011$.

9. (MH 9-10 2008) In the hexadecimal number system, what is $1A + 2E$?
   (a) 26
   (b) 38
   (c) 48
   (d) 72

   Solution 1. First we add the units digits: $A_{16} + E_{16} = 10 + 14 = 24 = 1 \cdot 16 + 8 = 18_{16}$, so
   $1A_{16} + 2E_{16} = 48_{16}$.
   Solution 2. Convert the given numbers to base 10, add, and convert back to base 16:
   $1A_{16} + 2E_{16} = (1 \cdot 16 + 10) + (2 \cdot 16 + 14) = 26 + 46 = 72 = 4 \cdot 16 + 8 = 48_{16}$.

10. (MH 9-10 2005) Find the numbers A, B, C, and D in the following base 6 addition.

$$\begin{array}{rcccc} & 3 & A & B & 3 \\ + & & 2 & 5 & C \\ \hline & D & 0 & 0 & 2 \end{array}$$

(a) $A = 1, B = 2, C = 3, D = 4$

(b) $A = 3, B = 0, C = 5, D = 3$

(c) $A = 3, B = 0, C = 5, D = 4$

(d) none of the above

Solution. We will work from right to left. First we will find C: we must have $3_6 + C_6 = 12_6$, so $3 + C = 8$, i.e. $C = 5$. Now we look at the next digit: we must have $B_6 + 5_6 = 5_6$, so $B = 0$. Next, $A_6 + 2_6 = 5_6$, so $A = 3$. Finally, since $3303_6 + 255_6 = 4002_6$, we have $D = 4$. So the answer is (c).

11. (MH 9-10 2003) $43_{Ten} = \_\_\_\_\__{Negative\ Ten}$

(a) 136

(b) 163

(c) 631

(d) none of the above

Solution. We need to write 43 in the form

$$c_k \cdot b^k + c_{k-1} \cdot b^{k-1} + \ldots + c_2 \cdot b^2 + c_1 \cdot b^1 + c_0 \cdot b^0$$

where $b = -10$. We have: $43 = 100 - 60 + 3 = 1 \cdot (-10)^2 + 6 \cdot (-10) + 3 \cdot (-10)^0 = 163_{-10}$.

12. (MH 9-10 2008) If the number 86 in base ten is represented as 321 in base b, then the number 123 in base b can be represented in base ten by what number?

(a) 12

(b) 25

(c) 35

(d) 38

Solution. First we need to find b such that $86 = 3 \cdot b^2 + 2 \cdot b + 1$. Solving this quadratic equation, we get two roots: $b = 5$ and $b = -17/3$. Since the base must be an integer, $b = 5$. Then $123_5 = 1 \cdot 25 + 2 \cdot 5 + 3 = 38$.

13. (MH 11-12 2008) Assume that b and c are two integers that are greater than one. In base b, c^2 is written as 10. Then b^2, when written in base c is

(a) 100

(b) 101

(c) 10000

(d) 1010

(e) It cannot be determined

Solution. If c^2 is written as 10 in base b, then $c^2 = b$. Then $b^2 = c^4$, so $b^2 = 10000_c$.

Decimals

14. (MH 9-10 2008) The number 0.125 (base 10) is represented by which of the following base 2 fractions?
    (a) 0.001_2
    (b) 0.01_2
    (c) 0.1_2
    (d) None of the above

    Solution. $0.125 = \frac{1}{8} = 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2^2} + 1 \cdot \frac{1}{2^3} = 0.001_2$.

15. (LF 9-12 2002) Suppose b is a positive integer base that satisfies the equation $(.111\ldots)_7 = (.222\ldots)_b$ (where the subscript indicates the base in the representation). Then $b =$
    (a) 14
    (b) 13
    (c) 6
    (d) 8
    (e) None of these

    Solution. $(.111\ldots)_7 = (.222\ldots)_b$ is equivalent to

$$\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots = 2 \cdot \frac{1}{b} + 2 \cdot \frac{1}{b^2} + 2 \cdot \frac{1}{b^3} + \ldots$$
$$\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots = 2\left(\frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \ldots\right)$$
$$\frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{2 \cdot \frac{1}{b}}{1 - \frac{1}{b}}$$
$$\frac{1}{6} = \frac{2}{b-1}$$
$$b - 1 = 12$$
$$b = 13$$

16. (LF 9-12 2008) The base-2 number (repeated decimal) $.\overline{01}_2 = .010101\ldots_2$ is equal to
    (a) $\frac{1}{3}$
    (b) $\frac{1}{4}$
    (c) $\frac{1}{5}$
    (d) $\frac{1}{6}$
    (e) None of the above

    Solution.

$$\begin{aligned} 0.\overline{01}_2 &= 0 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2^2} + 0 \cdot \tfrac{1}{2^3} + 1 \cdot \tfrac{1}{2^4} + 0 \cdot \tfrac{1}{2^5} + 1 \cdot \tfrac{1}{2^6} + \ldots \\ &= \tfrac{1}{4} + \tfrac{1}{4^2} + \tfrac{1}{4^3} + \ldots \\ &= \tfrac{\frac{1}{4}}{1 - \frac{1}{4}} \\ &= \tfrac{1}{4-1} \\ &= \tfrac{1}{3} \end{aligned}$$

17. (LF 9-12 2005) When converted to base 10, the infinite repeating base 3 number $0.\overline{12}_3$ is equal to

    (a) $\frac{1}{2}$
    (b) $\frac{4}{9}$
    (c) $\frac{5}{8}$
    (d) $\frac{5}{9}$
    (e) None of the above

    Solution.

$$\begin{aligned} 0.\overline{12}_3 &= 1\cdot\frac{1}{3}+2\cdot\frac{1}{3^2}+1\cdot\frac{1}{3^3}+2\cdot\frac{1}{3^4}+\dots \\ &= \left(\frac{1}{3}+\frac{1}{3^3}+\dots\right)+2\left(\frac{1}{3^2}+\frac{1}{3^4}+\dots\right) \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3^2}}+\frac{\frac{2}{3^2}}{1-\frac{1}{3^2}} \\ &= \frac{3}{9-1}+\frac{2}{9-1} \\ &= \frac{5}{8} \end{aligned}$$

18. (LF 9-12 2006) Let $(0.xyxyxy\dots)_b$ and $(0.yxyxyx\dots)_b$ be the base b representations of the two numbers A and B respectively, where x and y represent base b digits, not both of which are zero. Then $\frac{A}{B}=$

    (a) $\frac{y+b}{x+b}$
    (b) $\frac{x+b}{y+b}$
    (c) $\frac{xb+y}{yb+x}$
    (d) $\frac{yb+x}{xb+y}$
    (e) None of the above

    Solution.

$$\begin{aligned} A &= (0.xyxyxy\dots)_b \\ &= x\cdot\frac{1}{b}+y\cdot\frac{1}{b^2}+x\cdot\frac{1}{b^3}+y\cdot\frac{1}{b^4}+\dots \\ &= x\left(\frac{1}{b}+\frac{1}{b^3}+\dots\right)+y\left(\frac{1}{b^2}+\frac{1}{b^4}+\dots\right) \\ &= \frac{\frac{x}{b}}{1-\frac{1}{b^2}}+\frac{\frac{y}{b^2}}{1-\frac{1}{b^2}} \\ &= \frac{xb}{b^2-1}+\frac{y}{b^2-1} \\ &= \frac{xb+y}{b^2-1} \end{aligned}.$$

    Similarly,

$$\begin{aligned} B &= (0.yxyxyx\dots)_b \\ &= \frac{yb+x}{b^2-1} \end{aligned}.$$

    Then $\dfrac{A}{B}=\dfrac{\frac{xb+y}{b^2-1}}{\frac{yb+x}{b^2-1}}=\dfrac{xb+y}{yb+x}$.