Hoover High School Math League

March 25-26, 2009

Coordinate Geometry: Solutions

1. (MH 11-12 1997) Find the distance between $(2,-1)$ and $(7,4)$.

   (a) 6

   (b) $\sqrt{50}$

   (c) $\sqrt{130}$

   (d) 12

   (e) None of the above

   Solution. Using the distance formula

   $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

   we have $d = \sqrt{(7-2)^2 + (4+1)^2} = \sqrt{50}$.

2. (MH 11-12 2000) The equation of the line parallel to the line $4y - x = 20$ and containing the point $(2,-3)$ is

   (a) $y = 4x - 7$

   (b) $y = \frac{1}{4}x - \frac{7}{2}$

   (c) $y = \frac{3}{4}x + \frac{7}{2}$

   (d) $y = -4x + 5$

   (e) None of the above

   Solution. The equation of the given line can be rewritten as $y = \frac{1}{4}x + 5$, so its slope is $\frac{1}{4}$. Parallel lines have equal slopes, thus an equation of the required line, using the point-slope form

   $$y - y_0 = m(x - x_0),$$

   can be written as $y + 3 = \frac{1}{4}(x-2)$. This equation can be simplified to $y = \frac{1}{4}x - \frac{7}{2}$.

3. (MH 11-12 1997) Which of the following statements describes the graph of $f(x) = x^2 - 18x - 1$?

   (a) parabola with vertex $(-9, 242)$

   (b) parabola with vertex $(9, -82)$

   (c) parabola with vertex $(0,0)$

   (d) not a parabola

   (e) None of the above

   Solution. Since

   $$\begin{aligned} f(x) &= x^2 - 18x - 1 \\ &= x^2 - 18x + 81 - 82 \\ &= (x-9)^2 - 82, \end{aligned}$$

   its graph is a parabola with vertex $(9, -82)$.

4. (MH 11-12 2000) The graph of an equation $x^2+y^2+4y=14x+11$ is
   (a) a circle
   (b) a point
   (c) an ellipse
   (d) a parabola
   (e) None of the above

   Solution. The given equation can be rewritten as
   $x^2-14x+y^2+4y=11$
   $x^2-14x+49+y^2+4y+4=11+49+4$
   $(x-7)^2+(y+2)^2=8^2$
   This is an equation of a circle.

5. (MH 11-12 2000) The equation of a circle is $x^2+y^2+8x-2y+15=0$.
   (a) The center is $(-4,1)$ and the radius is $\sqrt{2}$.
   (b) The center is $(7,-2)$ and the radius is 8.
   (c) The center is $(4,3)$ and the radius is $\sqrt{5}$.
   (d) The center is $(-7,1)$ and the radius is 9.
   (e) None of the above

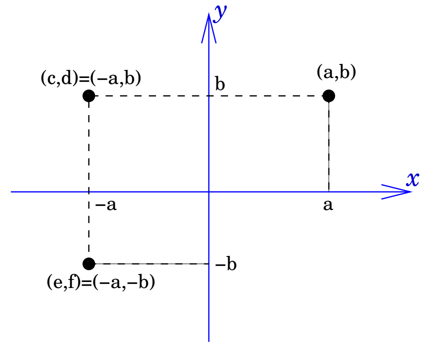
   Solution. The given equation can be rewritten as
   $x^2+8x+y^2-2y=-15$
   $x^2+8x+16+y^2-2y+1=-15+16+1$
   $(x+4)^2+(y-1)^2=2$
   $(x-(-4))^2+(y-1)^2=(\sqrt{2})^2$
   This is an equation of a circle with center at $(-4,1)$ and radius $\sqrt{2}$.

6. (MH 9-10 2002) The x-intercept of $3y-3x-8=0$ is
   (a) $\frac{8}{3}$
   (b) $\frac{3}{8}$
   (c) $\frac{-8}{3}$
   (d) $\frac{-3}{8}$

   Solution. When $y=0$, $-3x-8=0$. Solving for x gives $x=\frac{-8}{3}$.

7. (MH 9-10 2005) The point (a,b) is reflected over the y-axis to the point (c,d) which is reflected over the x-axis to the point (e,f). What is $ab-ef$?
   (a) 2
   (b) $2ab$
   (c) 0
   (d) none of the above

   Solution. When a point is reflected over the y-axis, its x-coordinate changes sign and its y-coordinate does not change. So $c=-a$ and $d=b$. When a point is reflected over the x-axis, its x-coordinate does not change and its y-coordinate changes sign. So $e=c=-a$ and $f=-d=-b$.



Then $ab - ef = ab - (-a)(-b) = ab - ab = 0$.

8. (MH 11-12 2006) Three vertices of parallelogram $ABCD$ are $A(-1,1)$, $B(4,5)$, and $C(3,1)$. Find the coordinates of the fourth vertex D.

   (a) $(-3,-4)$

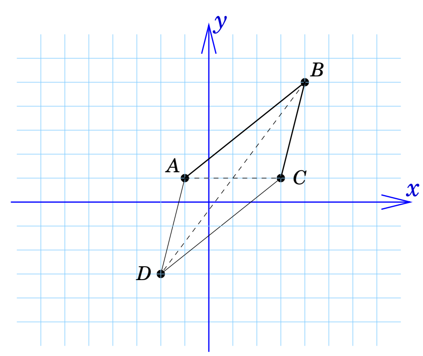
   (b) $(-2,-3)$

   (c) $(1,1)$

   (d) $(7,0)$

   (e) None of the above

   Solution 1. The center of the parallelogram is the midpoint of its diagonal AC, i.e. $\left(\frac{-1+3}{2}, \frac{1+1}{2}\right) = (1,1)$. Let (x,y) be the coordinates of point D. Since $(1,1)$ is also the midpoint of diagonal BC, $\left(\frac{4+x}{2}, \frac{5+y}{2}\right) = (1,1)$. So $\frac{4+x}{2} = 1$ and $\frac{5+y}{2} = 1$. Therefore $x = -2$ and $y = -3$.
   Solution 2. Draw a picture:



9. (MH 9-10 2005) A man travels 2 miles north, 2 miles east, one mile south, one mile west, 3 miles north, and 3 miles east. How far is he from the starting point?

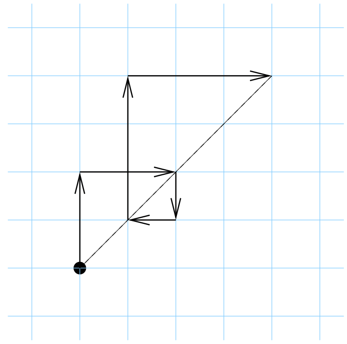
   (a) $2\sqrt{4}$ miles

   (b) 6 miles

   (c) $4\sqrt{2}$ miles

   (d) none of the above

   Solution. Draw a picture:



We see that the man is 4 miles north and 4 miles east of the starting point. The distance is $\sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$ miles.

10. (MH 11-12 2008) A line through the points $(m,-9)$ and $(7,m)$ has slope m. What is the value of m?

    (a) 1

    (b) 2

    (c) 3

    (d) 4

    (e) 5

    Solution. Using the slope formula, we have $\frac{m+9}{7-m} = m$. Then
    $m+9 = m(7-m)$
    $m+9 = 7m-m^2$
    $m^2-6m+9 = 0$
    $(m-3)^2 = 0$
    $m = 3.$

11. (MH 11-12 2005) Determine the point(s) of intersection (if any) of the line $x+y=2$ with the curve defined by $x^2-y^2=4$.

    (a) The line does not intersect the curve.

    (b) The line intersects the curve at the two points: $(2,0)$ and $(0,-2)$.

    (c) The line intersects the curve at the two points: $(-2,0)$ and $(2,0)$.

    (d) The line intersects the curve at the one point: $(2,0)$.

    (e) None of the above.

    Solution. Solving the equation $x+y=2$ for y and substituting into the second equation, we have
    $y = 2-x$
    $x^2-(2-x)^2 = 4$
    $x^2-(4-4x+x^2) = 4$
    $x^2-4+4x-x^2 = 4$
    $4x = 8$
    $x = 2$
    $y = 0$
    So there is only one intersection point: $(2,0)$.

12. (MH 11-12 2005) Given a line segment with endpoints $(-3,4)$ and $(-12,16)$, determine the coordinates of a point on the line whose distance from the right endpoint is one-third the length of the line segment.

(a) $(2,9)$

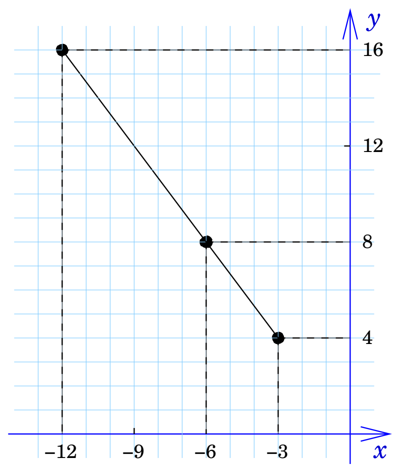
(b) $(-3+\sqrt{41}, 4+\sqrt{41})$

(c) $(-6,8)$

(d) $(5,-20/3)$

(e) None of the above

Solution. Let (x,y) be the coordinates of the point we are looking for. Then x is between -12 and -3 and such that the distance between x and -3 is one-third of the distance between -12 and -3. So $x=-6$. Similarly for y: it is between 4 and 16 such that the distance between y and 4 is one-third of the distance between 4 and 16. So $y=8$. Thus the answer is $(-6,8)$.



13. (MH 11-12 2000) If the function $f(x)=ax^2+3x-8$ has a minimum value at $x=-2$, then

(a) $a=5$

(b) $a=\frac{7}{9}$

(c) $a=\frac{3}{4}$

(d) $a=12$

(e) None of the above

Solution. In general, a quadratic function $f(x)=ax^2+bx+c$ has a minimum/maximum value at $x=-\frac{b}{2a}$. So in this case we have $-\frac{3}{2a}=-2$. Then $3=4a$, so $a=\frac{3}{4}$.

14. (MH 11-12 2000) What is the oblique asymptote of $f(x)=\frac{3x^2-7x+2}{x-2}$?

(a) $y=3x+1$

(b) $y=2x+1$

(c) $y=3x-2$

(d) $y=3x-1$

(e) None of the above

Solution. Factoring the numerator, we get $f(x)=\frac{3x^2-7x+2}{x-2}=\frac{(x-2)(3x-1)}{x-2}=3x-1$ when $x\neq 2$. So the graph of $f(x)$ is the line $y=3x-1$ except the point at $x=2$ (the function is undefined at $x=2$). Its oblique asymptote is $y=3x-1$.

15. (MH 11-12 2005) Determine the equation of the circle centered at $(-1,1)$ and tangent to the line $y=5$.

(a) $(x-1)^2+(y+1)^2=16$

(b) $(x-1)^2+(y+1)^2=25$

(c) $(x-1)^2+(y+1)^2=36$

(d) $(x+1)^2+(y-1)^2=16$

(e) $(x+1)^2+(y-1)^2=36$

Solution. Since the y-coordinate of the center is 1 and the circle is tangent to the line $y=5$, its radius is 4 (the distance between the center and the tangent line). Then its equation is $(x+1)^2+(y-1)^2=16$.

16. (MH 11-12 2005) Which of the following is the equation of a parabola with a maximum at $(-1,2)$ and passing through $(2,-1)$?

(a) $(y-2)=-\frac{1}{3}(x+1)^2$

(b) $(y+2)=(x-1)^2$

(c) $(y+2)=-\frac{1}{3}(x+1)^2$

(d) $(y-2)=-3(x+1)^2$

(e) $(y-2)=3(x+1)^2$

Solution. A parabola with a maximum at $(-1,2)$ has equation $y-2=a(x+1)^2$. Since it must pass through $(2,-1)$, it follows that $-1-2=a(2+1)^2$. Then $-3=9a$, so $a=-\frac{1}{3}$. Thus its equation is $y-2=-\frac{1}{3}(x+1)^2$.

17. (MH 11-12 1997) The graph of $y^2=2x^2+5x-3$ is:

(a) symmetric about the y-axis

(b) symmetric about the x-axis

(c) symmetric about the origin

(d) is not symmetric about any line

(e) None of the above

Solution. Since $y^2=2x^2+5x-3$ is equivalent to $(-y)^2=2x^2+5x-3$, for every point (x,y) that satisfies the equation, the point $(x,-y)$ also satisfies the equation, and vice versa. Therefore the graph is symmetric about the x-axis. (However, $y^2=2x^2+5x-3$ is not equivalent to $y^2=2(-x)^2+5(-x)-3$, so the graph is not symmetric about the y-axis. Also, $y^2=2x^2+5x-3$ is not equivalent to $(-y)2=2(-x)^2+5(-x)-3$, so the graph is not symmetric about the origin.)

18. (LF 9-12 2000) The slope of the line that goes through the point $(2,0)$ and is tangent to the circle $x^2+y^2=1$ in the first quadrant is

(a) $\frac{-1}{3}$

(b) $\frac{-1}{2}$

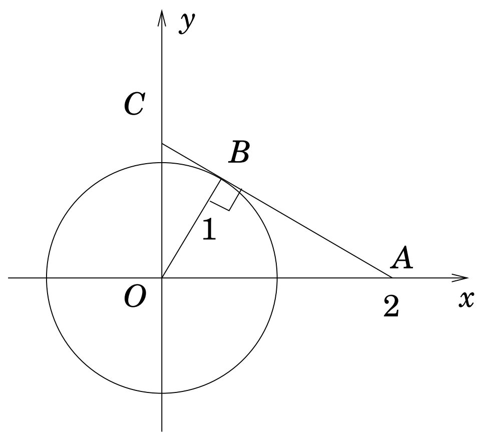
(c) $\frac{-1}{\sqrt{3}}$

(d) $\frac{-1}{\sqrt{2}}$

(e) None of these

Solution 1. Let (a,b) be the common point of the line and the circle. Then the slope of the line is $m_1=\frac{b-0}{a-2}=\frac{b}{a-2}$. The slope of the radius drawn to the point (a,b) is $m_2=\frac{b}{a}$. Since the line and the radius must be perpendicular, $m_1\cdot m_2=-1$. Thus $\frac{b}{a-2}\cdot\frac{b}{a}=-1$, so $b^2=-a(a-2)$.

Also, the point (a,b) satisfies the equation of the circle, so $a^2+b^2=1$. Substituting $b^2=-a(a-2)$ into this equation gives
$a^2-a(a-2)=1$
$a^2-a^2+2a=1$
$2a=1$
$a=\frac{1}{2}$.
Then $b^2=-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)=\frac{3}{4}$, so $b=\frac{\sqrt{3}}{2}$.
Therefore the slope of the line is $m_1=\frac{b}{a-2}=\frac{\frac{\sqrt{3}}{2}}{-\frac{3}{2}}=\frac{-1}{\sqrt{3}}$.
Solution 2. Since the tangent line (AC) and the radius (OB) are perpendicular, OAB is a right triangle.



Since $|OB|=1$ and $|OA|=2$, $\angle A=30^0$. So OAC is also a right triangle in which $\angle A=30^0$ and $|OA|=2$, therefore $|OC|=\frac{|OA|}{\sqrt{3}}=\frac{2}{\sqrt{3}}$. The slope of the line AC is then $-\frac{\frac{2}{\sqrt{3}}}{2}=\frac{-1}{\sqrt{3}}$.
Solution 3. Let the slope of the tangent line be m. Then its equation (using the point-slope form) is $y=m(x-2)$. The system

$$\begin{cases} y=m(x-2) \\ x^2+y^2=1 \end{cases}$$

must have only one solution. Substituting the y-value from first equation into the second, we get
$x^2+(m(x-2))^2=1$
$x^2+m^2x^2-4m^2x+4m^2=1$
$(m^2+1)x^2-4m^2x+(4m^2-1)=0$.
The quadratic equation has exactly one solution when the discriminant is 0:
$(-4m^2)^2-4(m^2+1)(4m^2-1)=0$
$16m^4-4(4m^4+3m^2-1)=0$
$16m^4-16m^4-12m^2+4=0$
$-12m^2+4=0$
$-3m^2+1=0$
$3m^2=1$
$m=\pm\frac{1}{\sqrt{3}}$.
However, if $m=\frac{1}{\sqrt{3}}$, then the common point of the line and the circle is in quadrant IV. So the answer is $m=-\frac{1}{\sqrt{3}}$.

19. (MH 11-12 2008) The graphs of the lines $y=x-2$ and $y=mx+3$ intersect at a point whose x-coordinate and y-coordinate are both positive if and only if

   (a) $m<1$

(b) $m = 1$

(c) $-\frac{3}{2} < m < 0$

(d) $-\frac{3}{2} < m$

(e) $-\frac{3}{2} < m < 1$

Solution. To find the intersection points, we set the right hand sides of the two equations equal: $x - 2 = mx + 3$, so $(m-1)x = -5$, thus $x = \frac{-5}{m-1}$. Then $y = x - 2 = \frac{-5}{m-1} - 2 = \frac{-5-2m+2}{m-1} = \frac{-2m-3}{m-1}$. Now, $x > 0$ if $\frac{-5}{m-1} > 0$, thus $m - 1 < 0$, so $m < 1$. Also, $y > 0$ if $\frac{-2m-3}{m-1} > 0$, thus $-2m - 3 < 0$, so $m > -\frac{3}{2}$. Thus the answer is $-\frac{3}{2} < m < 1$.

20. (LF 9-12 2004) Among all real number pairs (x, y) that satisfy $x^2 + x + y^2 + y = 1$, find the largest possible value of $x + y$.

(a) $\sqrt{2} - 1$

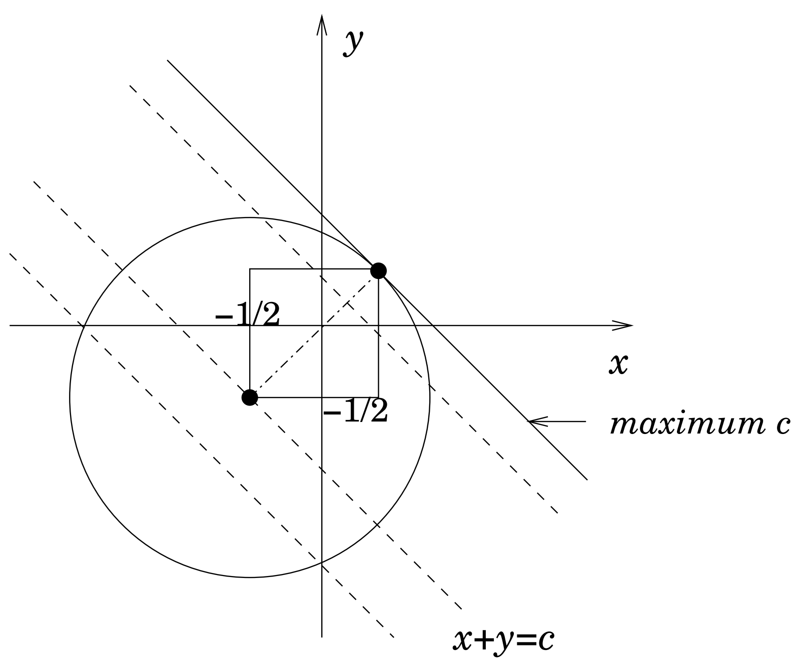
(b) 1

(c) $\sqrt{3} - 1$

(d) $\sqrt{3}$

(e) None of these

Solution. The equation $x^2 + x + y^2 + y = 1$ can be rewritten as
$x^2 + x + \frac{1}{4} + y^2 + y + \frac{1}{4} = \frac{3}{2}$
$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2.$
This is an equation of the circle with center at $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{3}}{\sqrt{2}}$.
The lines $x + y = c$ (where c is a constant) have slope -1.



The line with the largest possible value of c that has a common point with the circle is the tangent line l. The common point has coordinates $\left(-\frac{1}{2} + \frac{r}{\sqrt{2}}, -\frac{1}{2} + \frac{r}{\sqrt{2}}\right) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}, -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$, so the equation of the line is $x + y = -1 + \sqrt{3}$.

21. (MH 11-12 2000) For the ellipse $4x^2+9y^2-16x+18y-11=0$

(a) The center is $(2,-1)$ and the foci are $(2\pm\sqrt{5},-1)$

(b) The center is $(4,1)$ and the foci are $(2\pm\sqrt{5},-1)$

(c) The center is $(2,-1)$ and the foci are $(3\pm\sqrt{6},-1)$

(d) The center is $(4,9)$ and the foci are $(3\pm\sqrt{6},-1)$

(e) None of the above

Solution. The equation can be rewritten as
$4x^2-16x+9y^2+18y=11$
$4x^2-16x+16+9y^2+18y+9=11+16+9$
$4(x-2)^2+9(y+1)^2=36$
$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{2^2}=1$
So the center is $(2,-1)$ and principal axes are $a=3$ and $b=2$. Therefore the distance between the center and each of the foci is $c=\sqrt{a^2-b^2}=\sqrt{5}$. Then the foci are $(2\pm\sqrt{5},-1)$.

22. (LF 9-12 2004) Suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be inscribed in the diamond shape whose vertices are $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. Then $a^2+b^2=$

(a) 1

(b) a^2b^2

(c) $\frac{1}{ab}$

(d) a^4b^4

(e) None of the above

Solution. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is inscribed in the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$ if and only if the ellipse and the line that passes through $(1,0)$ and $(0,1)$ have exactly one common point. This line has equation $y=1-x$, and substituting this into the equation of the ellipse we get:
$\frac{x^2}{a^2}+\frac{(1-x)^2}{b^2}=1$
$b^2x^2+a^2(1-x)^2=a^2b^2$
$(a^2+b^2)x^2-2a^2x+(a^2-a^2b^2)=0.$
This quadratic equation has exactly one root if the discriminant is 0:
$(-2a^2)^2-4(a^2+b^2)(a^2-a^2b^2)=0$
$4a^4-4(a^4-a^4b^2+a^2b^2-a^2b^4)=0$
$4a^4-4a^4+4a^4b^2-4a^2b^2+4a^2b^4=0$
$4a^4b^2-4a^2b^2+4a^2b^4=0$
$a^4b^2-a^2b^2+a^2b^4=0$
$a^2b^2(a^2-1+b^2)=0.$
This equation implies that either $a=0$ (which is impossible) or $b=0$ (which is also impossible) or $a^2+b^2-1=0$, so $a^2+b^2=1$.

23. (MH 11-12 2005) Determine the equation in rectangular coordinates of $\cos\theta+\sin\theta=1$.

(a) $x=0$

(b) $y=0$

(c) $xy=0$

(d) The equation cannot be converted to rectangular coordinates

(e) None of the above

Solution. Multiplying both sides of the equation by r we get:
$r\cos\theta + r\sin\theta = r$
$x + y = \sqrt{x^2+y^2}$
$(x+y)^2 = x^2+y^2$
$x^2+2xy+y^2 = x^2+y^2$
$2xy = 0$
$xy = 0$
$x = 0$ or $y = 0$
However, we must have $x \geq 0$ and $y \geq 0$ in order for $x+y \geq 0$ to hold (it must hold because $\sqrt{x^2+y^2} \geq 0$). So we get the system

$$\begin{cases} xy = 0 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

(the solution set consists of all points on the positive half of the x-axis and all points on the positive half of the y-axis. This set cannot be described by a single equation in rectangular coordinates.)

24. (MH 11-12 2000) Convert the polar equation $r - r\sin\theta = 2$ to a rectangular equation.

(a) $x^2 - 4y + 4 = 0$

(b) $x^2 + 4y + 4 = 0$

(c) $x^2 - 2y - 2 = 0$

(d) $x^2 - 4y - 4 = 0$

(e) None of the above

Solution. The equation can be rewritten as
$r = r\sin\theta + 2$
$r^2 = (r\sin\theta + 2)^2$
$x^2 + y^2 = (r\sin\theta)^2 + 4r\sin\theta + 4$
$x^2 + y^2 = y^2 + 4y + 4$
$x^2 = 4y + 4$
$x^2 - 4y - 4 = 0$

25. (MH 11-12 2005) Planet M orbits around its sun, S, in an elliptical orbit with the sun at one focus. When M is closest to S, it is 2 million miles away. When M is farthest from S, it is 18 million miles away. Determine the equation of motion of planet M around its sun S, using S as the center of the coordinate plane and assuming the other focus lies on the positive x-axis.

(a) $\frac{x^2}{100} + \frac{y^2}{36} = 1$

(b) $\frac{x^2}{100} + \frac{y^2}{64} = 1$

(c) $\frac{(x-6)^2}{100} + \frac{y^2}{64} = 1$

(d) $\frac{(x-8)^2}{100} + \frac{y^2}{36} = 1$

(e) $\frac{(x-8)^2}{100} + \frac{(y-6)^2}{36} = 1$

Solution. Let a and b be principal axes and let c be the distance between the center of the ellipse and each of its foci. Then $2a = 2 + 18 = 20$, so $a = 10$. Also, $2c = 18 - 2 = 16$, so $c = 8$. Then $b = \sqrt{a^2 - c^2} = \sqrt{100 - 64} = 6$. If one of the foci is the origin of the coordinate plane and the other focus lies on the positive x-axis, then the center of the ellipse is at $(c, 0)$, i.e. $(8, 0)$. The equation of the ellipse is then

$\frac{(x-8)^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{(x-8)^2}{100} + \frac{y^2}{36} = 1$