Hoover High School Math League

April 15-16, 2009

Counting and Probability — Solutions

1. (MH 11-12 1997) At a sandwich shop there are 2 kinds of bread, 5 kinds of cold cuts, 3 kinds of cheese, and 2 kinds of dressing. How many different sandwiches can be prepared using one kind each of bread, cold cuts, cheese, and dressing?

   (a) 12

   (b) 50

   (c) 42

   (d) 30

   (e) None of the above

   Solution. $2 \cdot 5 \cdot 3 \cdot 2 = 60$.

2. (MH 11-12 1997) How many different six-digit numbers can be formed from the set $\{1,2,3,4,5,6,7,8,9\}$ if the digits may be repeated?

   (a) 6×9

   (b) 6^9

   (c) 9^6

   (d) $6+9$

   (e) None of the above

   Solution. Since there are 9 choices for each digit, there are $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^6$ possible different numbers.

3. (MH 9-10 2005) An integer between 1 and 1000 (inclusive) is selected at random. Find the probability that the integer is divisible by 5.

   (a) $\frac{1}{10}$

   (b) $\frac{1}{100}$

   (c) $\frac{1}{5}$

   (d) $\frac{1}{1000}$

   Solution. Since every fifth integer is divisible by 5, the probability is $\frac{1}{5}$.

4. (MH 11-12 1997) In how many ways can a committee of 4 persons be chosen from a group of 9 persons?

   (a) 9^4

   (b) $\frac{9!}{5!}$

   (c) 126

   (d) 36

   (e) None of the above

Solution. There are $\binom{9}{4} = \frac{9!}{4!5!} = \frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4} = 126$ ways to choose 4 persons out of 9.

5. (MH 11-12 2005) Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
   (a) 400
   (b) 380
   (c) 200
   (d) 190
   (e) None of the above

   Solution. There are $\binom{20}{2} = \frac{20\cdot19}{2} = 190$ ways to choose 2 persons out of 20, i.e. there are 190 pairs, so 190 handshakes.

6. (MH 11-12 2005) Three separate awards are to be presented to selected students from a class of 20. How many different outcomes are possible if a student can receive any number of awards?
   (a) 8,000
   (b) 1,140
   (c) 120
   (d) 60
   (e) 27

   Solution. Since there are 20 possible outcomes for each award, there are $20\cdot20\cdot20 = 8,000$ possible outcomes total.

7. (MH 11-12 2004) Someone simultaneously flips two coins. What is the probability of seeing exactly two presidents?
   (a) 12%
   (b) 25%
   (c) 30%
   (d) 50%
   (e) None of the above

   Solution. For each coin, the probability of the president is $\frac{1}{2}$. Therefore for two coins the probability is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = 25\%$.

8. (MH 11-12 2004) What is the probability of flipping a penny three times and seeing at least one head?
   (a) 25%
   (b) 50%
   (c) 75%
   (d) 87.5%
   (e) None of the above

   Solution. The probability of seeing no heads is $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$. So the probability of seeing at least one head $1-\frac{1}{8} = \frac{7}{8} = 87.5\%$.

9. (MH 11-12 2000) Twelve points are taken on a circle. How many triangles can be made using these points as vertices?

   (a) 110

   (b) 342

   (c) 352

   (d) 345

   (e) None of the above

   Solution. Since any thee points can be used, there are $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{6} = 220$ ways to choose 3 vertices out of 12 points.

10. (MH 11-12 1997) A bag contains 4 red, 2 white, and 3 blue marbles. Two marbles are drawn at random all at once. What is the probability that both are white?

   (a) $\frac{2}{9}$

   (b) $\frac{1}{6}$

   (c) $\frac{1}{36}$

   (d) $\frac{4}{9}$

   (e) None of the above

   Solution. There are $\binom{9}{2} = 36$ ways to choose 2 marbles out of 9. In only one case they are both white (because there are only two white marbles). Thus the probability is $\frac{1}{36}$.

11. (MH 11-12 2005) From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

   (a) 350

   (b) 792

   (c) 900

   (d) 3,991,680

   (e) None of the above

   Solution. There are $\binom{5}{2} = 10$ ways to choose 2 women out of 5, and $\binom{7}{3} = 35$ ways to choose 3 men out of 7, thus there are $10 \cdot 35 = 350$ different committees possible.

12. (MH 11-12 2004) What percentage of the first 1,000 natural numbers have a 2 or a 3 somewhere in them? (Choose the best answer)

   (a) 24%

   (b) 37%

   (c) 49%

   (d) 58%

   (e) None of the above

   Solution. It is easier to count the number of numbers that do not have 2 or 3 in them. So the available digits are 1, 4, 5, 6, 7, 8, 9, 0. Since there are 8 choices for each digit, there are 8^3 such numbers (one-digit and two-digit numbers can be written as three-digit numbers by adding one or two zeros in front). So $1000 - 8^3 = 1000 - 512 = 488$ numbers (out of 1000), or approximately 49%, have a 2 or a

3.

13. (MH 11-12 2004) What is the probability of rolling two fair dice and having the sum exceed 5?

    (a) $\approx 20\%$

    (b) $\approx 43\%$

    (c) $\approx 66\%$

    (d) $\approx 83\%$

    (e) None of the above

    Solution. Since there are 6 possible outcomes for each die, there are $6 \cdot 6 = 36$ possible outcomes total. It is faster to count the outcomes in which the sum does not exceed 5: $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$, $(3,2)$, $(4,1)$ – there are 10 such outcomes. Thus in 26 out of 36 outcomes the sum exceeds 5. The probability is $\frac{26}{36} = \frac{13}{18} \approx 0.722$, or approximately 72% (so the correct answer is "none of the above").

14. (MH 11-12 2005) In the game of craps, if a player rolls either a 7 or an 11 the first time s/he rolls the two dice, the player wins. What is the probability of the player winning on the first roll of the dice?

    (a) $\frac{2}{9}$

    (b) $\frac{1}{9}$

    (c) $\frac{1}{3}$

    (d) $\frac{1}{6}$

    (e) $\frac{1}{2}$

    Solution. As in the previous problem, there are 36 possible outcomes. Let's count the ones in which the sum is either 7 or 11: $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, $(6,1)$, $(5,6)$, $(6,5)$ – there are 8 such outcomes. Thus the probability is $\frac{8}{36} = \frac{2}{9}$.

15. (MH 11-12 2004) You roll three fair dice. What is the probability that you will see at least one "tripple" (1-1-1, 2-2-2, 3-3-3, and so on)?

    (a) $\approx 1\%$

    (b) $\approx 3\%$

    (c) $\approx 5\%$

    (d) $\approx 15\%$

    (e) None of the above

    Solution. There are 6^3 possible outcomes total, and in 6 of them the three numbers are all equal. So the probability is $\frac{6}{6^3} = \frac{1}{36} \approx 3\%$.

16. (MH 9-10 2005) The first digit of a seven-digit telephone number can never be a 0. How many telephone numbers have increasing digits?

    (a) 28

    (b) 720

    (c) 36

    (d) none of the above

    Solution. The question is equivalent to the following one: how many ways are there to choose 7 digits out of 9 (once we choose the digits, there is only one way to put them in the increasing order). The answer is $\binom{9}{7} = \frac{9!}{7!2!} = \frac{9 \cdot 8}{2} = 36$.

17. (MH 11-12 2004) From a regular deck of 52 playing cards, you turn over a 7 and then 8. What is the probability that the next card you turn over will be a face card (i.e., J, Q, or K)?

    (a) 12%

    (b) 18%

    (c) 24%

    (d) 30%

    (e) None of the above

    Solution. After a 7 and an 8 are turned over, there are 50 cards left, and 12 of them are face cards. So the probability of the next card being a face card is $\frac{12}{50} = 24\%$.

18. (MH 11-12 2004) Suppose you deal two cards from a regular deck of 52 cards. What is the probability that they will all be face cards?

    (a) $\approx 1\%$

    (b) $\approx 3\%$

    (c) $\approx 5\%$

    (d) $\approx 7\%$

    (e) None of the above

    Solution. There are $\binom{52}{2} = \frac{52 \cdot 51}{2} = 26 \cdot 51$ ways to choose 2 cards out of 52. There are $\binom{12}{2} = \frac{12 \cdot 11}{2} = 6 \cdot 11$ ways to choose 2 cards out of 12 face cards. Thus the probability that both cards will be face cards is $\frac{6 \cdot 11}{26 \cdot 51} = \frac{66}{26 \cdot 51} \approx \frac{63}{25 \cdot 50} = \frac{63 \cdot 8}{100 \cdot 100} = \frac{504}{100 \cdot 100} \approx \frac{5}{100} = 5\%$.

19. (MH 11-12 2000) Two cards are randomly selected from an ordinary 52-card playing deck. What is the probability that they form a blackjack? In other words, what is the probability that one of the cards is an Ace and the other one is either a ten, a jack, a queen, or a king?

    (a) $\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}$

    (b) $\frac{\binom{4}{1}\binom{4}{1}}{\binom{52}{2}}$

    (c) $\frac{2\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}$

    (d) $\frac{2\binom{4}{1}\binom{4}{1}}{\binom{52}{2}}$

    (e) $\frac{\binom{13}{1}\binom{52}{1}}{\binom{52}{2}}$

    Solution. There are $\binom{52}{2}$ ways to choose 2 cards out of 52. Now let's count the number of blackjacks. There $\binom{4}{1}$ ways to choose an Ace and $\binom{16}{1}$ ways to choose the other card, so there are $\binom{4}{1}\binom{16}{1}$ different blackjacks. Thus the probability of a blackjack is $\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}$.

20. (MH 11-12 2004) Someone simultaneously flips three coins. What is the probability of seeing exactly two presidents?

   (a) 25%

   (b) 37.5%

   (c) 50%

   (d) 62.5%

   (e) None of the above

   Solution. There are $\binom{3}{2} = 3$ ways to choose two coins (the ones that show presidents) out of 3, and the probability that these two show presidents and the third coin does not is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, thus the probability of seeing exactly two presidents is $\frac{3}{8} = 37.5\%$.

21. (MH 11-12 2004) Someone simultaneously flips four coins. What is the probability of seeing exactly two presidents?

   (a) 25%

   (b) 37.5%

   (c) 50%

   (d) 62.5%

   (e) None of the above

   Solution. This problem is similar to the previous one, but here there are $\binom{4}{2} = 6$ ways to choose two coins (the ones that show presidents) out of 4, and the probability that these two show presidents and the other two coins do not $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$, thus the probability of seeing exactly two presidents is $\frac{6}{16} = \frac{3}{8} = 37.5\%$.

22. (MH 11-12 2005) A pressure control apparatus contains three electronic tubes. The apparatus will not work unless all tubes are operative. If the probability of failure of each tube over some time interval is 0.1, what is the probability of failure of the apparatus?

   (a) 99.9%

   (b) 72.9%

   (c) 27.1%

   (d) 0.1%

   (e) None of the above

   Solution. The probability that all tubes will remain operative is $0.9 \cdot 0.9 \cdot 0.9 = 0.729$, therefore the probability that at least one tube fails is $1 - 0.729 = 0.271 = 27.1\%$.

23. (MH 9-10 2005) John counted the number of subsets that a set X has, and Mary counted the number of subsets the Y has. If John counted 96 more subsets than Mary, how many elements does X have?

   (a) 5

   (b) 9

   (c) 3

   (d) none of the above

   Solution. Let x be the number of elements of the set X. Then the number of subsets of the set X is 2^x

(because each element has two choices: to be or not to be an element of a subset). Similarly, let y be the number of elements of the set Y, then the number of subsets of the set Y is 2^y.
So $2^x - 2^y = 96$.
$2^y(2^{x-y} - 1) = 2^5 \cdot 3$
Since 2^{x-y} is odd and 2^y cannot not have any odd factors, $2^y = 2^5$ and $2^{x-y} - 1 = 3$.
So $y = 5$, $2^{x-y} - 1 = 3$
$2^{x-5} = 4$
$x - 5 = 2$
$x = 7$.

24. (MH 9-10 2005) In how many distinguishable ways can the letters of the word CINCINNATI be arranged?

    (a) 45,850

    (b) 36,520

    (c) 50,400

    (d) none of the above

    Solution. There are 10! ways to arrange 10 letters. However, the letter C appears twice, and it does not matter which of the two C's we use first, and which second. Thus we double counted each arrangement, so we have to divide the number of arrangements by 2. Similarly, there are 3 letters N, and there are 6 possible arrangements of 3 letters, so we divide by 6. Finally, since there are 3 letters I, we divide by 6 again. So the number of distinguishable ways that the letters of CINCINNATI be arranged in is $\frac{10!}{2 \cdot 6 \cdot 6} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 6 \cdot 6} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 5 \cdot 2 = 50400$.

25. (MH 11-12 1997) Find the number of distinguishable permutations of letters in the word TALLAHASSEE.

    (a) 6!

    (b) 2!

    (c) $\frac{11!}{9!}$

    (d) $\frac{11!}{3!2!2!2!}$

    (e) None of the above

    Solution. This problem is similar to the previous one. Since there are 11 letters, but A appears three times, L appears twice, S appears twice, and E appears twice, the number of permutations here is $\frac{11!}{3! \cdot 2! \cdot 2! \cdot 2!}$.

26. (MH 11-12 1997) How many nine-digit social security numbers are there containing three 5's, two 6's, and four 8's?

    (a) $\frac{9!}{5!6!8!}$

    (b) $\frac{9!}{3!2!4!}$

    (c) 9!

    (d) $9 \times 3 \times 2 \times 4$

    (e) None of the above

    Solution. The idea is similar to that of the previous two problems. The number of permutations of three 5's, two 6's, and four 8's is $\frac{9!}{3!2!4!}$.

27. (MH 9-10 2005) The number $2^7 \times 3^4 \times 5 \times 7^2 \times 11^3$ is divisible by how many perfect squares?

(a) 36

(b) 48

(c) 60

(d) none of the above

Solution. The prime factorization of a perfect square contains each prime factor an even number of times. A number is a divisor of $2^7 \times 3^4 \times 5 \times 7^2 \times 11^3$ if its prime factorization contains seven or less 2's, four or less 3's, one or no 5's, two or less 7's, three or less 11's, and no other factors. Combining these two conditions, we see that in the prime factorization of a perfect square that is a divisor of $2^7 \times 3^4 \times 5 \times 7^2 \times 11^3$, the factor 2 has 4 choices (not appear at all, appear twice, or 4 or 6 times); the factor 3 has 3 choices (not appear at all, or appear twice or 4 times); the factors 7 and 11 each have 2 choices (not appear at all or appear twice); no other factor may appear. So the total number of possibilities is $4 \cdot 3 \cdot 2 \cdot 2 = 48$.