Hoover High School Math League

Proportional Reasoning, Ratios, and Rates   Solutions

Ratios of quantities, proportions

1. (MH 9-10 2005) The ratio of x to y is $5 : 3$. The ratio of y to z is $9 : 10$. What is the ratio of x to z?

   (a) $3 : 2$

   (b) $2 : 1$

   (c) $3 : 1$

   (d) none of the above

   Solution. If $\frac{x}{y} = \frac{5}{3}$ and $\frac{y}{z} = \frac{9}{10}$, then $\frac{x}{z} = \frac{xy}{yz} = \frac{x}{y} \cdot \frac{y}{z} = \frac{5}{3} \cdot \frac{9}{10} = \frac{5 \cdot 9}{3 \cdot 10} = \frac{1 \cdot 3}{1 \cdot 2} = \frac{3}{2}$. So the answer is $3 : 2$.

2. (MH 9-10 2005) If y is inversely proportional to the square of x and $y = 5$ when $x = 5$, what is y when $x = \frac{1}{10}$?

   (a) 2150

   (b) 250

   (c) 1250

   (d) none of the above

   Solution. If y is inversely proportional to the square of x, then $y = \frac{c}{x^2}$ for some constant c. We are given that $y = 5$ when $x = 5$, so $5 = \frac{c}{5^2}$. This implies $c = 125$. So $y = \frac{125}{x^2}$.
   If $x = \frac{1}{10}$, then $y = \frac{125}{\left(\frac{1}{10}\right)^2} = 125 \cdot 10^2 = 12500$.

3. (MH 9-10 1998) Suppose x varies directly as the square of y and inversely as the product of z and w, and suppose $x = 10$ when $y = 5$, $z = 2$, and $w = 10$. Find x if $y = 10$, $z = 5$, and $w = 2$.

   (a) 100

   (b) 80

   (c) 50

   (d) 20

   Solution. If x varies directly as the square of y and inversely as the product of z and w, then $x = \frac{cy^2}{zw}$ for some constant c. We are given that $x = 10$ when $y = 5$, $z = 2$, and $w = 10$, so $10 = \frac{c \cdot 5^2}{2 \cdot 10}$. Simplifying, we get
   $10 = \frac{25c}{20}$
   $200 = 25c$
   $c = 8$.
   So $x = \frac{8y^2}{zw}$. Then if $y = 10$, $z = 5$, and $w = 2$, we have $x = \frac{8 \cdot 10^2}{5 \cdot 2} = 80$.

4. (MH 9-10 2003) Find the ratio of A to B if $\frac{2B+6A}{3A-B} = 10$.

   (a) 1:3

   (b) 3:1

   (c) 1:2

(d) none of the above

Solution 1. Multiplying both sides of the given equation by $3A - B$ and simplifying we get:

$2B + 6A = 10(3A - B)$

$2B + 6A = 30A - 10B$

$12B = 24A$

$\frac{12}{24} = \frac{A}{B}$

So the ratio of A to B is 1:2.

Solution 2. Let $\frac{A}{B} = x$. Dividing both the numerator and the denominator of the given equation by B we get:

$\frac{2 + 6\frac{A}{B}}{3\frac{A}{B} - 1} = 10$

$\frac{2 + 6x}{3x - 1} = 10$

$2 + 6x = 10(3x - 1)$

$2 + 6x = 30x - 10$

$12 = 24x$

$x = \frac{1}{2}$.

5. (MH 9-10 1998) The distance a body falls from rest varies directly to the square of the time it falls. If an object falls 144 feet in 3 seconds, how far will it fall in 6 seconds?

    (a) 567 ft

    (b) 576 ft

    (c) 32 ft

    (d) none of the above

Solution. Let d denote the distance and let t denote the time. Then we are given that $d = ct^2$ for some constant c and that $144 = c \cdot 3^2$, so $c = 16$. Thus $d = 16t^2$. When $t = 6$, we have $d = 16 \cdot 6^2 = 576$.

6. (MH 9-10 2006) If 26 blops weigh as much as 4 glorps and 2 gloops, while 8 blops and 2 glorps have the same weight as 2 gloops, how many blops have the same weigh as 3 gloops?

    (a) 10

    (b) 12

    (c) 21

    (d) None of the above

Solution. Let x be the weight of one blop, let y be the weight of one glorp, and let z be the weight of one gloop. Then we have

$26x = 4y + 2z, \ 8x + 2y = 2z$. We need to know how much times x is equal to $3z$. First let's divide both equations by 2:

$13x = 2y + z, \ 4x + y = z$. Then

$y = z - 4x$, and $13x = 2(z - 4x) + z$. Then

$13x = 2z - 8x + z$

$21x = 3z$.

So the answer is 21.

<div align="center">Parts</div>

7. (MH 9-10 1998) The number 216 is divided into 3 parts proportional to 2, 3, and 4. Find the smallest

<div align="center">2</div>

part.

(a) 24

(b) 48

(c) 12

(d) 60

Solution. Let the three parts be $2x$, $3x$, and $4x$. Then
$2x + 3x + 4x = 216$
$9x = 216$
$x = 24.$
So the smallest part is $2 \cdot 24 = 48$.

8. (MH 11-12 2005) A painter needs to make a certain shade of green paint. She has 7.5 ounces of yellow paint, and she can make this shade of green by mixing 8 parts blue with 3 parts yelow paint. How much blue paint does she need?

(a) 60 ounces

(b) 30 ounces

(c) 22.5 ounces

(d) 20 ounces

(e) None of the above

Solution. Let x be the amount (in ounces) of the blue paint needed. Then $\frac{x}{7.5} = \frac{8}{3}$
$3x = 8 \cdot 7.5$
$3x = 60$
$x = 20.$

Speed

9. (MH 9-10 2002) Mr. Jones traveled for 2 hours at 40 mph and then for another 3 hours at 80 mph. What is his average rate over the entire trip?

(a) 57

(b) 60

(c) 64

(d) none of the above

Solution. Mr. Jones covered the distance 80 miles in 2 hours and then 240 miles in the next 3 hours. So the total distance is 320 miles and the total time is 5 hours. The average speed is then $\frac{320}{5} = 64$ mph.

10. (MH 11-12 2006) A boat travels 145 km downstream and it takes it the same time to travel 95 km upstream. The speed of the current is 5 km/h. Determine the speed of the boat in still water.

(a) 22 km/h

(b) 23 km/h

(c) 24 km/h

(d) 25 km/h

(e) None of the above

Solution. Let the speed of the boat in still water be x, then the speed of the boat going downstream is $x + 5$ and the speed of the boat going upstream is $x - 5$. It would take $\frac{145}{x+5}$ hours to travel 145 km downstream and it would take $\frac{95}{x-5}$ hours to travel 95 km upstream. Thus

$\frac{145}{x+5} = \frac{95}{x-5}$
$145(x - 5) = 95(x + 5)$
$29(x - 5) = 19(x + 5)$
$29x - 145 = 19x + 95$
$10x = 240$
$x = 24.$

Other rates

11. (MH 9-10 2006) If 7 dogs can eat 7 bowls of dog food in 7 minutes, how long will it take 14 dogs to eat 14 bowls of dog food?

    (a) 28 minutes

    (b) 21 minutes

    (c) 14 minutes

    (d) 7 minutes

    Solution. If 7 dogs can eat 7 bowls of dog food in 7 minutes, then 1 dog eats 1 bowl in 7 minutes. So 14 dogs eat 14 bowls in the same 14 minutes.

12. (MH 9-10 2005) Nine woodchucks can chuck eight pieces of wood in three hours. How much wood can a woodchuck chuck in one hour?

    (a) 8/3 pieces

    (b) 8/9 pieces

    (c) 8/27 pieces

    (d) none of these

    Solution. If nine woodchucks can chuck eight pieces of wood in three hours, then each woodchuck chucks $\frac{8}{9}$ pieces of wood in three hours, thus it chucks $\frac{8}{27}$ pieces in one hour.

13. (MH 11-12 2006) Kate and Helen have to make 60 pyramids for their geometry project. Kate can make them in 6 hours, and Helen can make them in 12 hours. Assuming that they will work at these rates, how long will it take them to make 60 pyramids together?

    (a) 3 hours

    (b) 4 hours

    (c) 4 hours 30 minutes

    (d) 5 hours

    (e) None of the above

    Solution. If Kate can make 60 pyramids in 6 hours, her rate is 10 pyramids per hour. Similarly, if Helen can make 60 pyramids in 12 hours, her rate is 5 pyramids per hour. Working together, then can make 15 pyramids per hour. So it will take them 4 hours to make 60 pyramids.

14. (MH 11-12 2003) It takes 5 hr 20 min for Bill to type some text. It takes 4 hr 40 min for John to type the same text. One day they typed 90 pages together. How many pages did Bill type and how many pages did John type on that day?

(a) 42 and 48 pages

(b) 40 and 50 pages

(c) 30 and 60 pages

(d) 38 and 52 pages

(e) 44 and 46 pages

Solution 1. Let Bill's rate be x pages per hour and let John's rate be y pages per hour. Then
$5\frac{1}{3} \cdot x = 4\frac{2}{3} \cdot y$
$\frac{16}{3} \cdot x = \frac{14}{3} \cdot y$
$16x = 14y$
$8x = 7y$
$y = \frac{8}{7}x.$
When they are typing together, their rate is $x+y$. So it took them $\frac{90}{x+y}$ hours to type 90 pages. In this time, Bill typed $\frac{90}{x+y} \cdot x$ pages and John typed $90 - \frac{90}{x+y} \cdot x$ pages. Now let's calculate these values:
$\frac{90}{x+y} \cdot x = \frac{90x}{x + \frac{8}{7}x} = \frac{90}{1 + \frac{8}{7}} = \frac{90}{\frac{15}{7}} = \frac{90 \cdot 7}{15} = 6 \cdot 7 = 42,$
$90 - 42 = 48.$

Solution 2. Let's start the same way as in Solution 1, until we get to $8x = 7y$. So $\frac{x}{y} = \frac{7}{8}$. This means that of every 15 pages they type together, Bill types 7 and John types 8. so If they typed $90 = 6 \cdot 15$ pages together, then of those Bill typed $6 \times 7 = 42$ and John typed $6 \times 8 = 48$ pages.

Geometry: similar triangles, etc.

15. (MH 9-10 2003) Find the area of the square *ROSE* which is inscribed in right triangle *ADI*, if $AE = 12$ and $SI = 27$.
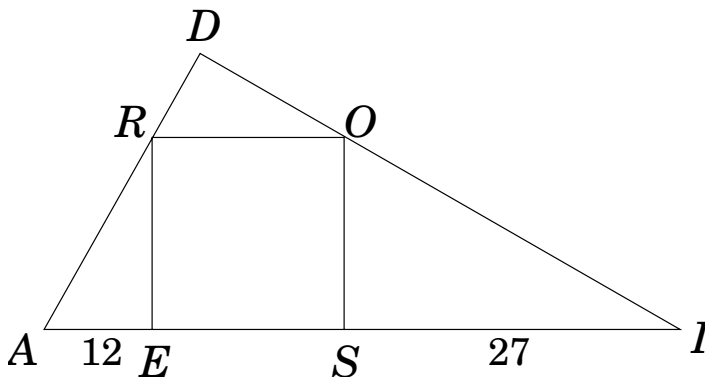
(a) 729

(b) 144

(c) 324

(d) none of the above

Solution. First notice that triangles *AER* and *ISO* are similar:
$\angle I = 90^0 - \angle A$ since *ADI* is a right triangle;
in triagnle *AER*, $\angle R = 90^0 - \angle A$;
so in *AER* and *ISO*, $\angle I = \angle R$ and $\angle S = 90^0 = \angle E$.
In similar triangles the corresponding sides are proportional:
$\frac{|AE|}{|ER|} = \frac{|SO|}{|IS|}.$
Let x denote the length of the side of square *ROSE*. Then $\frac{12}{x} = \frac{x}{27}.$

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It follows that $x^2 = 12 \cdot 27 = 324$, so the area of the square is 324.

16. (MH 9-10 2005) In the figure below, AB is perpendicular to DE and AC is perpendicular to CB. $AE = 6$, $EB = 7$, and $BC = 5$. What is the area of the quadrilateral $EBCD$?
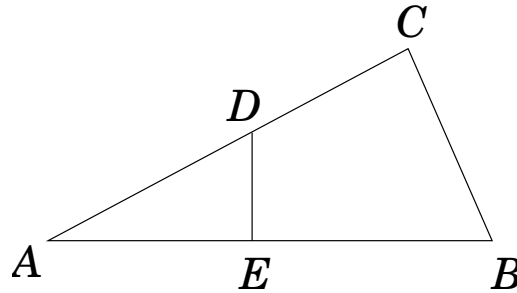
(a) 12

(b) 45

(c) 22.5

(d) none of the above

Solution. To find the area of the quadrilateral $EBCD$, we will subtract the area of triangle AED from the area of triangle ACB.

In triangle ACB, we know $BC = 5$ and $AB = AE + EB = 6 + 7 = 13$. Then by Pythagorean theorem, $AC^2 = AB^2 - BC^2 = 13^2 - 5^2 = 169 - 25 = 144$, so $AC = 12$. Thus the area of triangle ACB is $\frac{1}{2} \cdot 12 \cdot 6 = 30$.

In triangle AED we know $AE = 6$ and we need to find ED in order to find the area. Since triangles AED and ACB are similar,
$$\frac{ED}{AE} = \frac{BC}{AC}$$
$$\frac{ED}{6} = \frac{5}{12}$$
$$ED = 2.5.$$
Then the area of triangle AED is $\frac{1}{2} \cdot 6 \cdot 2.5 = 7.5$. Finally, the area of the quadrilateral $EBCD$ is $30 - 7.5 = 22.5$.

17. (MH 11-12 2008) On a dark lonely night, a man is standing 5 yards away from a street light. The man is 2 yards tall and the light is 6 yards high. How long is the man's shadow?

(a) 2 yards

(b) 4 yards

(c) $\frac{5}{2}$ yards

(d) $\frac{7}{3}$ yards

(e) None of the above

Solution. Let the length of the man's shadow be x. Using the two similar triangles, we get:
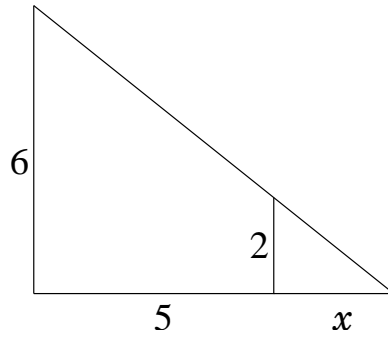
$\frac{6}{5+x} = \frac{2}{x}$
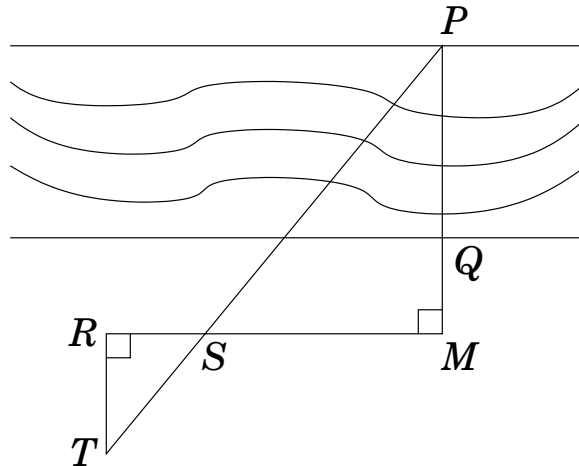
$6x = 2(5+x)$

$6x = 2x+10$

$4x = 10$

$x = 2.5.$

18. (MH 9-10 2006) Suppose you wish to measure the distance across the river shown below. You find that the distance *SM* is 35 feet, *RS* is 8 feet, *RT* is 16 feet, and *QM* is 10 feet. How far is it across the river?

(a) 40 feet

(b) 50 feet

(c) 70 feet

(d) 60 feet

Solution. Triangles *SMP* and *SRT* are similar. Therefore

$\frac{MP}{MS} = \frac{RT}{RS}$

$\frac{MP}{35} = \frac{16}{8}$

$MP = 70.$

Since $QM = 10$, $PQ = 60$.